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THERMODYNAMIC MODEL OF CAPITAL EXTRACTION IN ECONOMIC SYSTEMS

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SUMMARY

In this paper the properties of the wealth function of an economic system are studied. An economic analog of the Gibbs-Duhem equation is derived. Equilibrium states and limiting profit extraction regimes in non-equilibrium economic systems are obtained for the Cobb-Douglas wealth function.

KEY WORDS

nonequilibrium and irreversible thermodynamics, economics, econophysics, financial markets, business and management

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INTRODUCTION

A *macro-system* is one that includes a large number of elements. It can only be controlled on a macro level by changing parameters averaged over the ensemble of its elements. Thermodynamic systems containing large numbers of molecules provide a classical, long-studied example of macro-systems. Economic systems containing large numbers of economic micro-agents provide a second important class of macro-systems. Further examples of macro-systems are given by migration systems, segregated systems, whose elements interact through a uniform single medium, etc.

An important feature of macro-systems is that direct contact between two macro-systems leads to a stochastic interaction between their elements on a micro-level. This occurs spontaneously and is irreversible, because it is necessary to supply the systems with external energy or capital to return them to their initial states.

Mathematical models of macro-systems can be divided into structural analytical models, which derive a system's behaviour from the behaviour and statistical properties of its micro-elements, and phenomenological models, which directly model macro behaviour. The macro-system (thermodynamic) approach to economics was developed by von Neumann, Samuelsen, Lihnerovich, Rozonoer, Martinás and others. A comprehensive list of references can be found in the reviews [1, 7] and the monographs [9, 10].

In this paper we will employ the following definitions:

1. The state of a macro- system is described by two types of variables – extensive and intensive. The former are proportional to the scale of the system, while the latter are independent of scale change. For example, in thermodynamics, volume, internal energy and mass are extensive while concentration is intensive. In economics, endowments of resources and capital are extensive, while resource prices are intensive. Extensive variables in an isolated macro-system obey balance equations. For systems with transformers such as chemical reactors or production companies, these balances govern the transformation of one type of extensive variables into another, for example.
2. We consider three types of sub-systems:
   2.1. Systems with infinite capacity and constant intensive variables (*reservoirs*). For example, heat reservoir in thermodynamics or market in economics where the trading flows are so large that the influence of an individual trader on prices is infinitely small and prices are constant(prices do not depend on the trading volume).
   2.2. *Finite capacity systems*, with intensive variables that depend on its extensive variables for fixed time scale. For example, the temperature of a thermodynamic system with finite heat capacity depends on its internal energy. For economic system with finite capacity resource's estimate depend on its endowment. We will also refer to a finite capacity economic system as economic system.
   2.3. *Active systems* with controllable intensive variables. For example, a working body of a heat engine or an economic intermediary that operates between economic systems.
3. Kinetics of exchange processes. The difference of intensive variables of two contacting macro-systems with finite capacities leads to an emergence of exchange flows. These flows in turn depend of the intensive variables of contacting systems and are directed in such a way that the values of intensive variables move closer. In equilibrium these values are the same and there are no flows.

Similarly to the approach adopted in finite-time thermodynamics ([9, 16, 17]) we assume that the economic system under consideration consists of subsystems in internal equilibrium and that all irreversibility is concentrated on the contact surfaces between these subsystems.

The maximal work problem of transforming a non-organized form of energy (heat, chemical energy) into an organized form (mechanical work, electric current, work of separation) plays the major role in thermodynamics. Its solution led to the introduction of exergy, the maximal amount of nonorganised energy that can be transformed into work. This measure does not take into account the rate of work (transformer's power). Accounting for this constraint led to the replacement of exergy with a more general notion of *work capacity* [12].

The problem of extracting the maximal capital from a macro-system, with subsystems that have different initial endowments of liquid capital and illiquid capital in the form of various resources, represents microeconomic analogy of thermodynamics' maximal work problem. In thermodynamics the work can be fully transformed into other forms of energy. Similarly all capital (money, basic resource) in microeconomics can be fully transformed into any other resources. Other resources can only be transformed into money if there is demand for it from the other economic system. We shall call the limiting amount of money that can be extracted from economic system subject to some conditions the *profitability* of this system. Production of work or extraction of capital is not possible unless the system includes active subsystems. In thermodynamics they are heat engines and other transformers, in economics they are economic intermediaries or production companies.

The special variable that gives a quantitative measure of irreversibility of system's processes plays a central role in macro-system's theory. When macrosystem approaches the equilibrium the value of this variable increases. In equilibrium it attains maximum. In thermodynamics this measure is called entropy. It has been proven that entropy is a function of extensive variables and is an extensive variable itself. Therefore the entropy is a homogeneous function of the degree one. In microeconomics equilibrium economic system is described by the wealth function S that depends on the stock of resources N and stock of capital N_0. Note that wealth function of a system that consists of a number of non-uniform subsystems is not additive. Furthermore, in the general case subsystems' wealth functions can have different dimensionality. The principle difference between thermodynamic and economic macro-systems is that in thermodynamics an exchange of only one type of material or energy is possible (heat transferred from a hot to a cold body). During this exchange the entropy of one of contacting subsystems can decrease but the entropy of the other will increase in such a way that the total system's entropy tends to maximum. In economics all exchanges are voluntary. Therefore the wealth function of each participant does not decrease. In most cases that is only possible for a multi-resources' exchange.

PROPERTIES OF WEALTH FUNCTION AND ANALOGY OF GIBBS-DUHEM EQUATION

The state of an economic system can be described by the vector of $N = (N_1; \ldots; N_n)$ resources and capital N_0. These are extensive variables. Economic system is prepared to sell resource N_i at a price that is not less than pi, and to buy it at a price that is not higher than p_i. We shall call p_i the equilibrium price estimate of the i-th resource by economic system. These estimates themselves are economic system's intensive variables. For a finite-capacity economic system p depends on N and N_0.

Suppose that active subsystem interacts with economic system by buying and selling its resources in such a way that the state of economic system changes cyclically and the exchange is executed at the equilibrium prices p. Then the increase of economic system's capital is

$$\mathrm{d}N_0 = -\oint p\mathrm{d}N \,. \tag{1}$$

If dN_0 was not equal zero than active subsystem would be able to extract arbitrary large profit by exchanging with one economic system without changing the state of the environment (if dN_0 < 0 then active subsystem extracts resource in direct cycle, if dN_0 > 0 then it does it in inverse cycle). This is not feasible in economics (Ville Axiom [14]) and therefore the integral (1) must be equal zero and a function $M(N)$ exists such that

$$p_M \mathrm{d}M = \sum_{i=1}^{n} p_i \mathrm{d}N_i \tag{2}$$

Let us construct function S, such that

$$\mathrm{d}S = p_0(\mathrm{d}N_0 + p_M \mathrm{d}M) = p_0\left(\mathrm{d}N_0 + \sum_{i=1}^{n} p_i \mathrm{d}N_i\right). \tag{3}$$

From Pfaffian forms theory it is known that for two variables N_0 and M there exists integrating multiplier p_0, such that dS is total differential.

The formal proof of the existence of the wealth function $S(N_0; N)$ in an economic system is similar to this sketch [2, 4, 10, 15]. The multiplier p_0 = dS/dN_0 is the estimate of the basic resource (capital) and the estimate of the i-th resource is

$$p_i = \frac{1}{p_0}\frac{\partial S}{\partial N_i}, \quad i = 1, \ldots, n. \tag{4}$$

Expressing dN_0 from (3) we get

$$\mathrm{d}N_0 = \frac{\mathrm{d}S}{p_0} - \sum_{i=1}^{n} p_i \mathrm{d}N_i \,. \tag{5}$$

The wealth function and all its arguments are proportional to the scale of the system. Therefore, it is a homogeneous function of the degree one. From Euler theorem it follows that it can be written in the following form

$$S(N_0, N) = p_0\left(\mathrm{d}N_0 + \sum_{i=1}^{n} p_i \mathrm{d}N_i\right) = \sum_{i=0}^{n} \frac{\partial S}{\partial N_i}\mathrm{d}N_i \tag{6}$$

The estimates of resources and capital $p_0(N)$; $p_i(N)$ here are homogeneous functions of the degree zero.

From (6) follows that

$$N_0 = \frac{S}{p_0} - \sum_{i=1}^{n} p_i N_i \,, \tag{7}$$

$$\mathrm{d}N_0 = \frac{\mathrm{d}S}{p_0} + S\cdot \mathrm{d}\left(\frac{1}{p_0}\right) - \sum_{i=1}^{n}(p_i \mathrm{d}N_i + N_i \mathrm{d}p_i)\,. \tag{8}$$

Comparison of equations (8) and (5) yields the following equation that links capital's estimate and resources' estimates

$$S\cdot \mathrm{d}\left(\frac{1}{p_0}\right) - \sum_{i=1}^{n} N_i \mathrm{d}p_i = 0\,. \tag{9}$$

Similarly comparing the differential S from (6) with the expression (3), we get

$$N_0 \mathrm{d}p_0 + \sum_{i=1}^{n} N_i \mathrm{d}(p_0 p_i) = 0\,. \tag{10}$$

The conditions (9) and (10) follow from the existence of function S and its homogeneity. They are economics analogies of Gibbs-Duhem equations. The following conditions also follow from the existence of function S:

$$\frac{\partial(p_0 p_i)}{\partial N_j} = \frac{\partial(p_0 p_j)}{\partial N_i} = \frac{\partial^2 S}{\partial N_i \partial N_j}, \tag{11}$$

$$\frac{\partial p_0}{\partial N_j} = \frac{\partial(p_0 p_j)}{\partial N_0} = \frac{\partial^2 S}{\partial N_0 \partial N_j}. \tag{12}$$

It is easy to see that from conditions (11), (12), it follows that

$$\frac{\partial p_i}{\partial N_j} + p_i \frac{\partial p_j}{\partial N_0} = \frac{\partial p_j}{\partial N_i} + p_j \frac{\partial p_i}{\partial N_0}, \quad i, j = 1, \ldots, n. \tag{13}$$

Conditions (11) and (12) are economic analogies of Maxwell equations.

One of the forms of wealth function that obeys the conditions (3 – 10), is the Cobb-Douglas function

$$S_k = A \prod_{i=0}^{n} N_i^{\gamma_i}, \tag{14}$$

with $A > 0$ a constant, $\gamma \geq 0$ and $\sum_{i=0}^{n} \gamma_i = 1$. An alternative form of S proposed by Martinás is

$$S_M = \sum_{i=1}^{n} g_i N_i \ln\left(\frac{N_0}{k_i N_i}\right), \tag{15}$$

where g_i and k_i are some constants.

For economic reservoir

$$S^0 = p^0\left(N_0^0 + \sum_{i=1}^{n} p_i^0 N_i^0\right), \tag{16}$$

where p_{00} and p_{i0} are constants. The dimensionality of function S is in units of currency of the corresponding economic system. The dimensionality of the estimates p_{i} is the amount of capital per unit of i-th resource.

Demand and supply functions are defined as dependencies of the amounts of resource sold (bought) on the price. If we consider its time-dependent version then demand and supply functions will describe the dependence of the flow of the traded resource on its price. This flow is equal zero if the price C_i is equal to the estimate p_i. The equations (9 – 13) show that estimates can not be arbitrary functions of resources' endowments. They must be homogeneous functions of zero degree that obey these equations.

This makes possible to model empirical data for nearly equilibrium economic systems. In [13] it was demonstrated on historic data for Sweden that the conditions (11) and (12) held during the periods when it was nearly equilibrium and broke down during economic crisis of 1930-th. Fulfilment of the conditions (11) and (12) guarantees the existence of a function S.

EQUILIBRIUM IN ECONOMIC SYSTEMS

We consider an economic system that has a wealth function and which includes m subsystems with given initial endowments of resources $N_\nu(0)$, $\nu = 1, ..., m$.

SYSTEM WITH ECONOMIC RESERVOIR

Economic reservoir corresponds to the perfect competition market with constant prices. They are determined by exogenous factors or by the conditions of non-accumulation of resource on the market.

The conditions of equilibrium in such systems is reduced to the equality of the resources' estimates in all subsystems to the market's prices

$$p_{i\nu}(\overline{N_\nu}) = p_i^0, \quad i = 1, \ldots, n, \nu = 1, \ldots, m. \qquad (17)$$

Here $\overline{N_\nu} = (\overline{N_0}, \overline{N_1}, \ldots)$ is vector of stocks of resources in equilibrium for ν-th subsystem. The balances of capital in each subsystem

$$\sum_{i=1}^{n}\left[\overline{N_{i\nu}} - N_{i\nu}(0)\right]p_i^0 = N_{0\nu}(0) - \overline{N_{0\nu}}, \quad \nu = 1, \ldots, m, \qquad (18)$$

are to be added to conditions (17). The system (17) and (18) determines $(n+1)m$ variables $\overline{N_{i\nu}}$, $i = 0, \ldots, n$, $\nu = 1, \ldots, m$.

Suppose that the wealth functions for each subsystem have Cobb-Douglas form (14)

$$S_\nu = A_\nu \prod_{i=0}^{n} N_{i\nu}^{\gamma_{i\nu}}, \quad \nu = 1, \ldots, m. \qquad (19)$$

The estimates then are

$$p_{0\nu} = S_\nu \frac{\gamma_{0\nu}}{N_{0\nu}}, \quad p_{i\nu} = \frac{\gamma_{i\nu} N_{0\nu}}{\gamma_{0\nu} N_{i\nu}} = S_\nu \frac{\gamma_{i\nu}}{p_{0\nu} N_{i\nu}}. \qquad (20)$$

Let us introduce the variable

$$V_\nu = N_{0\nu} + \sum_{i=1}^{n} p_i^0 N_{i\nu}, \quad \nu = 1, \ldots, m, \qquad (21)$$

called *capitalization* of the ν-th subsystem in terms of market prices. The condition (18) states that capitalization is constant during equilibrium exchange

$$\overline{V_\nu} = V_\nu(0) = V_\nu. \qquad (22)$$

The conditions (17) take the form

$$\frac{\gamma_{i\nu} \overline{N_{0\nu}}}{\gamma_{0\nu} \overline{N_{i\nu}}} = p_i^0, \quad \nu = 1, \ldots, m, i = 1, \ldots, n. \qquad (23)$$

The solution of the system (22) and (23) has the form

$$\overline{N_{i\nu}} = V_\nu \frac{\gamma_{i\nu}}{p_i^0}, \quad \overline{N_{0\nu}} = V_\nu \gamma_{0\nu}, \quad i = 1, \ldots, n, \nu = 1, \ldots, m. \qquad (24)$$

The corresponding equilibrium value of the wealth function is

$$\overline{S_\nu} = V_\nu \gamma_{0\nu}^{\gamma_{0\nu}} \prod_{i=1}^{n} \left(\frac{\gamma_{i\nu}}{p_i^0}\right)^{\gamma_{i\nu}}, \quad \nu = 1, \ldots, m. \qquad (25)$$

If the vector of market's prices p_0 is not determined by the external factors but is set at such level that all resources offered at the *auction* are sold then in addition to the conditions of equilibrium (17) the capital balance (18) and conditions of non-accumulation of resources on the market are needed

$$\sum_{\nu=1}^{n}\left[N_{i\nu}(0) - \overline{N_{i\nu}}\right] = 0, \quad i = 1, \ldots, n. \qquad (26)$$

These conditions determine n variables p_i^0.

If S_ν has the form (19) then the equations (26) take the form

$$\frac{1}{p_i^0}\sum_{\nu=1}^{m} V_\nu \gamma_{i\nu} = \sum_{\nu=1}^{m} N_{i\nu}(0), \quad i = 1, \ldots, n. \qquad (27)$$

We denote

$$\overline{N_{i\Sigma}} = \sum_{\nu=1}^{m} N_{i\nu}(0)\,, \qquad \overline{N_{ij}} = \sum_{\nu=1}^{m} \gamma_{i\nu} N_{j\nu}(0)\,,$$

and after taking into account (21) the conditions (27) can be rewritten as a linear system

$$\overline{N_{i0}} + \sum_{j=1}^{n} p_j^0 \overline{N_{ij}} - p_i^0 \overline{N_{i\Sigma}} = 0\,, \quad i = 1, \ldots, n, \tag{28}$$

that determines price vector p^0.

SYSTEMS WITHOUT ECONOMIC RESERVOIR

Since economic exchange is a voluntarily action by an agent, it is possible to exchange a resource if and only if this resource estimates for contacting systems have opposite signs. For example, production waste has negative estimate for one subsystem and positive for another, which have a capability to process it. If these estimates have the same sign in both contacting systems then only an exchange where at least two resources are traded can take place (flow of a resource and counter flow of capital and counter flow of another resource (barter)). It turns out that a state where vectors of estimates p for all subsystems are identical and any exchange that increases the wealth function of ν-th function

$$S_\nu = p_{0\nu}\left(N_{0\nu} + \sum_{i=1}^{n} p_i N_{i\nu} \right) = p_{0\nu} V_\nu\,, \tag{29}$$

reduces the wealth function of at least one other contacting subsystems, is an equilibrium state. That is, in economics (unlike thermodynamics) all Pareto-optimal states turned out to be equilibrium states. Some of these equilibrium states correspond to an exchange via an auction when the prices are determined by the conditions of non-accumulation (26). In this case capitalization V_ν of each subsystem in equilibrium is equal to the initial capitalization, which determines the equilibrium distribution of the basic resource.

If functions S_ν have all the same dimensionality (which is not always the case) then it is possible to single out the state in the Pareto set for which the value of the wealth function is maximal. This means that transfer into another equilibrium state would not give wealth function gains to some subsystem sufficient high to offset losses to wealth function of other subsystems.

It is clear that this maximal wealth function state corresponds to the equality of capital estimates

$$p_{0\nu} = p_0\,, \quad \nu = 1, \ldots, m. \tag{30}$$

which determines, jointly with conditions of equilibrium and conditions of non-accumulation, the distribution of all resources.

EXTRACTION OF CAPITAL

Consider a system with an active subsystem whose goal is to extract capital. For simplicity we assume that this subsystem is an intermediary, which resells resources without processing it.

UNCONSTRAINT EXCHANGE TIME

Extraction of capital from a system is only possible if the its initial state is non-equilibrium, that is, if vectors of estimates $p_\nu(0)$ for its subsystems are different. The process terminates in equilibrium when

$$p_{i\nu}(\overline{N_{0o}}, \overline{N_\nu}) = p_i^0\,, \quad i = 1, \ldots, n, \nu = 1, ..., m. \tag{31}$$

Maximum of the extracted capital corresponds to the minimum of the following expression

$$\sum_{\nu=1}^{m} \overline{N_{0\nu}} \to \min\,. \tag{32}$$

Intermediary buys resources at lowest prices (from subsystems with estimates of the i-th resource lower than p_i^0) and sells it at the highest prices (to subsystems with estimates higher than p_i^0). Both buying and selling are reversible with zero Increments of wealth function. The state of equilibrium is determined by $m\times n$ conditions (31), m reversibility conditions

$$S_\nu(\overline{N_{0o}},\overline{N_\nu}) = S_\nu(N_{0\nu}(0), N_\nu(0)) = S_\nu, \quad \nu = 1, ..., m, \qquad (33)$$

and condition of non-accumulation of resources by the intermediary

$$\sum_{\nu=1}^{n}\left[N_{i\nu}(0) - \overline{N_{i\nu}}\right] = 0, \quad i = 1, \ldots, n. \qquad (34)$$

The system (31), (33) and (34) determines $(m+1)n$ subsystems' state variables and n equilibrium estimates p_i. Naturally in equilibrium in a system with an intermediary N_0 and N are different from equilibrium during a direct exchange. Maximum of the extracted capital is

$$M = \sum_{\nu=1}^{m}\left[N_{0\nu}(0) - \overline{N_{0\nu}}\right]. \qquad (35)$$

For the Cobb-Douglas wealth function (19) the conditions of equilibrium take the form (23), where instead of condition of constant capitalisation (22) one needs to use the condition of constancy of S_ν

$$A_\nu \prod_{i=0}^{n} \overline{N_{i\nu}}^{\gamma_{i\nu}} = S_\nu(0), \quad \nu = 1, ..., m, \qquad (36)$$

jointly with equation (34).

DISSIPATION AND CAPITAL EXTRACTION IN A FINITE TIME IN A CLOSED ECONOMIC SYSTEM

If the duration of the process is finite and constraint then the increment of the wealth function and the amount of extracted capital depend on the demand and supply functions (that is, the dependencies of the flow rates of resources on the price differentials). When an intermediary buys resource from economic system in a finite time is has to increase the offered price above the equilibrium price. As a result it spends more capital. Similarly during a sale in a finite time an intermediary has to give a discount on the equilibrium price. This reduces its capital. The product of the flow between two EA on the difference between the prices of buying and selling describes the current losses of capital due to the factor of irreversibility capital dissipation)

$$\sigma(p,c) = g(p,c)(p-c). \qquad (37)$$

Capital dissipation measures irreversibility of the processes in the system.

Reciprocity conditions for flows that linearly depend on price differences

The causes of resource-exchange flows (their "driving forces") is the differential between resources' estimate by the economic systems and the price offered by an intermediary. Suppose that deviations from the equilibrium are small and the flows can be considered as linear functions of the price estimate differences.

The driving force for the i–th resource is $\Delta_i = p_i - c_i$. We denote the flow directed to economic system as positive. We get

$$g_i = \sum_{\nu=1}^{m} a_{\nu i}\Delta_\nu = \sum_{\nu=1}^{m} a_{\nu i}(p_\nu - c_\nu), \quad i = 1, \ldots, n. \qquad (38)$$

We shall call matrix A with elements $a_{i\nu}$ the matrix of kinetic coefficients of economic system. It determines kinetics of its exchange with environment. The flow of resource causes the counter-flow of capital such that

$$\frac{dN_0}{dt} = -\sum_{i=1}^{n} c_i g_i \ . \tag{39}$$

The change in the value of the wealth function here is

$$\frac{dS}{dt} = \frac{\partial S}{\partial N_0}\frac{dN_0}{dt} + \sum_{i=1}^{n}\frac{\partial S}{\partial N_i} g_i = -p_0 \sum_{i=1}^{n} c_i g_i + p_0 \sum_{i=1}^{n} p_i g_i =$$

$$= p_0 \sum_{i=1}^{n} (p_i - c_i) g_i = p_0 \Delta^T A \Delta \ . \tag{40}$$

Δ is the vector of driving forces.

Because capital's estimate $p_0 > 0$, resource exchange can be executed with buyer's and seller's consent and wealth function does not decrease, the matrix is positive. Let us show that it is also symmetrical. Indeed, if driving forces are expressed in terms of flows using equation (38) then for any infinitesimally short time period the expression (40) will take the form

$$\frac{dS}{dp_0} = dN^T B dN \ , \tag{41}$$

where dN is the vector-column of increases of resources' stocks, $B = A^{-1}$. The elements $b_{i\nu}$ of this matrix are

$$b_{i\nu} = \frac{\partial^2}{\partial N_i \partial N_\nu}\left(\frac{S}{p_0}\right) = b_{\nu i} \ , \quad i = 1, \ldots, n, \ \nu = 1, \ldots, m. \tag{42}$$

Thus, the matrix B is positive and symmetrical. Therefore its inverse demand/supply matrix is also symmetrical and positive near equilibrium. The following reciprocity relations hold: *the effect of the difference between price and estimate of the v-the resource on the flow of i-th resource is the same as the effect of the difference between the price and estimate of the i-th resource on the flow of the v-th.*

The optimal buying (selling) of resource for linear resource exchange

System with one finite-capacity economic subsystem. Consider a system with one finite capacity subsystem (economic system) and an active subsystem (intermediary). Suppose the initial and finite states of economic system are given $N(0) = (N_0(0), N_1(0), \ldots, N_m(0))$, $\overline{N} = (N_0(\tau), N_1(\tau), \ldots, N_m(\tau))$. The intermediary sets such vector of prices $c(t) = (c_1(t), c_2(t), \ldots, c_m(t))$, that the final capital of economic system $N_0(\tau)$ is minimal. First we assume that the flow depends linearly on the driving forces

$$g = A\Delta = A(c - p) \ . \tag{43}$$

Matrix A with $m \times m$ elements a_{ij} is positive and symmetrical, $(c - p)$ is the vector with elements

$$\Delta_i = c_i - p_i(N). \tag{44}$$

We denote $\delta_i = N_m(\tau) - Ni(0)$ and rewrite the problem as follows

$$\overline{N_0} = N_0(\tau) = N_0(0) + \int_0^\tau \sum_{i=1}^{m} c_i(t) \sum_{j=1}^{m} a_{ij}(c_j - p_j) dt \to \min_{c(t)} \ , \tag{45}$$

subject to constraints

$$\int_0^\tau g_i(t) dt = -\delta_i \ , \quad i = 1, \ldots, m. \tag{46}$$

The problem (44 – 46) corresponds to maximum of the extracted capital as $M = N_0(0) - N_0(\tau)$.

Let us express c in terms of g using (43)

$$c(g) = p + A^{-1} g = p + Bg \ . \tag{47}$$

The problem

$$\overline{N_0} = N_0(0) + \int_0^{\tau} g^T (p + Bg) dt \to \min_g , \tag{48}$$

subject to constraints (46) gives lower bound on the optimal solution of the problem (44 – 46), because the condition (45) has been deleted. If this solution is realisable, that is, if it obeys the condition (45), then the solution of the problem (48) and (46) is also a solution of (44 – 46).

Because matrix B is symmetrical and positive the problem (48) and (46) is a convex averaged problem of non-linear programming. Its optimal solution is constant and equal to

$$g_i^* = -\frac{\delta_i}{\tau} = \frac{N_i(0) - N_i(\tau)}{\tau}, \quad i = 1, \ldots, m. \tag{49}$$

The corresponding solution of the equations (45) is realisable ($N_i^*(t) \geq 0$)

$$N_i^*(t) = N_i(0) - \frac{N_i(0) - N_i(\tau)}{\tau} t, \quad i = 1, \ldots, m. \tag{50}$$

Substitution of this dependence into $p_i(N)$ will determine $p_i^*(t)$ and the equation (47) yields the optimal price $c^*(t)$.

System with a number of economic subsystems. Consider a system with n economic systems and an intermediary. Intermediary buys resource from some subsystems and sells it to others. The maximum of the extracted capital corresponds to its minimum in all economic systems at time τ. That is, the solution of the problem

$$\sum_{\nu=1}^{m} N_{\nu 0}(\tau) = \sum_{\nu=1}^{m} \left(N_{\nu 0} - \int_0^{\tau} \sum_{i=1}^{n} c_{i\nu}(t) g_{i\nu}(t,c) \mathrm{d}t \right) \to \min_c , \tag{51}$$

where the conditions (45) hold for each economic system, and condition (46) is replaced with the condition of non-accumulation of resources by the intermediary

$$\sum_{\nu=1}^{m} \int_0^{\tau} g_{i\nu}(t) dt = 0, \quad i = 1, \ldots, n. \tag{52}$$

The values of $\overline{N_{i\nu}}$ in this problem are free.

Because both buying and selling should proceed optimally, the flows of resource should be constant and must obey the conditions (49)

$$g_{i\nu}^* = \frac{N_{i\nu} - \overline{N_{i\nu}}}{\tau}, \quad i = 1, \ldots, n, \nu = 1, \ldots, m. \tag{53}$$

From (53) it follows that the criterion (51) is determined by the subsystems' final states

$$\overline{N_0} = \sum_{\nu=1}^{m} \overline{N_{\nu 0}}(\overline{N_\nu}) \to \min_{\overline{N_\nu}} . \tag{54}$$

$\overline{N_\nu}$ must be chosen in such a way that $\overline{N_0}$ is minimal subject to constraint (52), which takes the form

$$\sum_{\nu=1}^{m} \overline{N_{\nu i}} = \sum_{\nu=1}^{m} N_{\nu i}(0) = \overline{N}(0), \quad i = 1, \ldots, n. \tag{55}$$

The conditions of optimality (54) and (55) on $\overline{N_{\nu i}}$ for this problem take the form

$$\frac{\partial \overline{N_{\nu 0}}}{\partial \overline{N_{\nu i}}} = -\lambda_i , \quad i = 1, \ldots, n, \nu = 1, \ldots, m. \tag{56}$$

Since $\partial \overline{N_{\nu 0}} / \partial \overline{N_{\nu i}} = -c_{\nu i}(\tau)$, (56) is reduced to the condition that at time τ the buying and selling prices must be the same for all economic systems for each kind of resources $c_{\nu i}(\tau) =$

λ_i, $\forall\ \nu$. The conditions (55) and (56) consist of $n(1 + m)$ equations with respect to unknowns λ_i, $i = 1, \ldots, n$ and $\overline{N_{\nu i}}$, $i = 1, \ldots, n$, $\nu = 1, ..., m$. The dependencies $\overline{N_{\nu 0}}$ on $\overline{N_{\nu i}}$, in turn are determined by $g^*_{i\nu}(\overline{N_{\nu i}})$, $p_\nu(N)$ and by the matrix B via the equation (47).

Their substitution into (54) allows us to find the minimum of the residual capital and therefore, the maximum of the extracted capital.

Conditions of optimal trading for non-linear dependence of flows on price differences

Consider the problem of optimal buying (selling) of resource for non-linear resource-exchange law. This problem for scalar resource was considered in [11], [12]. We denote the amount of resource as ΔN and the duration of exchange as τ. The problem of optimal buying takes the form

$$\overline{N_0} = N_0(\tau) \to \min_{c(t)}, \tag{57}$$

subject to constraints

$$\int_0^\tau g(c, p(N_0, N))\mathrm{d}t = \Delta N, \tag{58}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -g(c, p(N_0, N)),\ N(0) = N^0, \tag{59}$$

$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = cg(c, p(N_0, N)),\ N_0(0) = N_0^0, \tag{60}$$

In this problem $c(t)$ is the price set by the intermediary, $p(N_0, N)$ is resource estimate by the subsystem, $g(c, p(N_0, N))$ is the flow of resource that depend on c and p in such a way that

$$\begin{aligned} &\mathrm{Sign}(g) = \mathrm{Sign}(c - p) \\ &g(c, p) = 0 \qquad c = p. \end{aligned} \tag{61}$$

The conditions of optimality for the problem (57-60) have the form ([11, 12])

$$\frac{\mathrm{d}}{\mathrm{d}N}\frac{\partial g/\partial c}{g^2(p,c)} = \frac{(\partial g/\partial p)(\partial p/\partial N_0)}{g^2(p,c)}. \tag{62}$$

In [11] and [12] it is also shown that criterion (57) is equivalent to criterion of minimal dissipation

$$\sigma = \int_0^\tau g(c, p)(c - p)\mathrm{d}t \to \min. \tag{63}$$

We consider the optimal buying problem for vector flows, where the flow of each i-th resource g_i ($i = 1, \ldots, n$) depends on the vector of prices $c = (c_1, \ldots, c_n)$ and estimates $p = (p_1, \ldots, p_n)$. Here the minimum of spent capital corresponds to the problem with criterion (57) subject to constraints

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$$\int_0^\tau g_i(c, p(N_0, N))\mathrm{d}t = \Delta N_i \,, \tag{64}$$

$$\frac{\mathrm{d}N_\mathrm{i}}{\mathrm{d}t} = -g_i(c, p(N_0, N)), \quad N_i(0) = N_i^0 \quad i = 1, \ ..., n \,, \tag{65}$$

$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = \sum_{i=1}^n g_i(c, p(N_0, N)), \quad N_0(0) = N_0^0 \,. \tag{66}$$

The maximal amount of residual capital decreases monotonically when τ increases, tending to reversible limit we already found. Indeed, if this dependence was not monotonic then the intermediary could stop exchange at $\tau_1 < \tau$ when this capital was minimum. This means that during an optimal process the r.h.s. of the equation (66) has one sign. This allows us to choose the new independent variable

$$\mathrm{d}t = \frac{\mathrm{d}N_0}{\sum_{i=1}^n c_i g_i(c, p)} \,, \tag{67}$$

and replace minimisation of the residual capital with minimisation of the duration of the process for given $\overline{N}_0$

$$\tau = \int_{N_0^0}^{\overline{N}_0} \frac{\mathrm{d}N_0}{\sum_{i=1}^n c_i g_i(c, p)} \to \min_{c(t)} \,, \tag{68}$$

subject to constraints

$$\int_{N_0^0}^{\overline{N}_0} \frac{\mathrm{d}N_i}{\mathrm{d}N_0} \mathrm{d}N_0 = -\int_{N_0^0}^{\overline{N}_0} \frac{g_i(c, p(N_0, N))\mathrm{d}N_0}{\sum_{i=1}^n c_i g_i(c, p)} = \overline{N}_i - N_i^0 = \delta_i \,, \tag{69}$$

$$\frac{\mathrm{d}N_\mathrm{i}}{\mathrm{d}N_0} = -\frac{g_i(c, p(N_0, N))}{\sum_{i=1}^n c_i g_i(c, p)}, \quad N_i(N_0^0) = N_i^0 \quad i = 1, \ ..., n \,. \tag{70}$$

We assume that the solution of the problem (68 – 70) is not degenerate ($\psi_0 = -1$) and denote the scalar product as follows

$$\sum_j c_j g_j = (c, g), \quad \sum_j \psi_j g_j = (\psi, g) \,.$$

The Hamiltonian function of this problem is

$$H = -\frac{1 + (\psi, g)}{(c, g)} \,, \tag{71}$$

$$\frac{\mathrm{d}N_\mathrm{i}}{\mathrm{d}N_0} = -\frac{g_i}{(c, g)}, \quad N_i(N_0^0) = N_i^0, \quad N_i(\overline{N}_0) = \overline{N}_i, \quad j, i = 1, \ ..., n \,. \tag{72}$$

The weak conditions of optimality here are

$$\frac{\partial H}{\partial c_j} = 0 \Rightarrow H(c, g, \psi) = \frac{\sum_i \psi_i (\partial g_i / \partial c_j)}{g_j + \sum_i c_i (\partial g_i / \partial c_j)} \,, \tag{73}$$

$$\frac{\partial \psi_i}{\partial N_0} = -\frac{\partial H}{\partial N_i} \Rightarrow \frac{\partial \psi_i}{\partial N_0} = -\frac{1}{(c, g)} \sum_j \left\{ \left[\psi_j - c_j H(c, g, \psi) \right] \frac{\partial g_j}{\partial N_i} \right\}, \quad i = 1, \ ..., n \,. \tag{74}$$

Conditions (74) show that for an optimal process the expression in the r.h.s. of this equation has the same value for all j. The boundary conditions for adjoint variables are determined by the boundary conditions $N(0) = N_0^0$ and $N(\tau) = \overline{N}$. $\overline{N}_0$ can be viewed as a parameter. This parameter can be determined from the condition that in the optimal process the integral (68) equals τ. Conditions (74) are a system of linear equations with respect to vector of adjoint variables ψ. After elimination of ψ from (73) the optimality conditions can be reduced to a form similar to (62).

MAXIMAL RATE OF PROFIT EXTRACTION IN OPEN ECONOMIC SYSTEM

STATIONARY STATE OF AN OPEN ECONOMIC SYSTEM WITH LINEAR RESOURCE-EXCHANGE LAWS, PRINCIPLE OF MINIMAL CAPITAL DISSIPATION

Consider an open microeconomic system shown in Fig. 1.

Suppose the system is in a stationary state; each of its n subsystems ($i = 1, \ldots, n$) exchanges m types of resources g_{ij}^{ν} ($i, j = 1, \ldots, n$; $\nu = 1, \ldots, m$) with other subsystems; the flows of resources depend linearly on the differences of estimates $\Delta_{ij}^{\nu} = \overline{p_j^{\nu}} - p_i^{\nu}$. For each subsystem and each resource these flows are constrained by the conditions of the balance

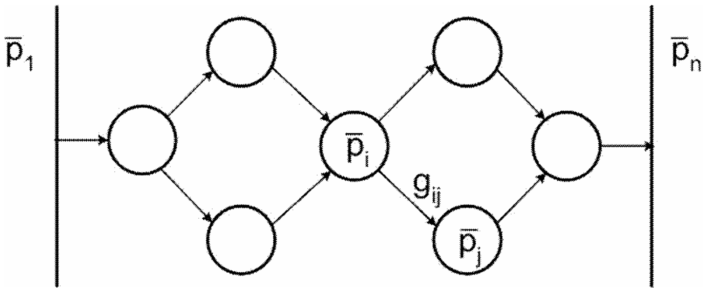

Figure 1. The structure of an open microeconomic system.

$$\sum_{j=1}^{n} g_{ij}^{\nu}(\Delta_{ij}) = 0, \quad i = 2, \ldots, n-1, \quad \nu = 1, \ldots, m. \tag{75}$$

Here Δ_{ij} is the vector of driving forces with components Δ_{ij}^{μ}.

If these flows depend linearly on the estimates' difference then (see (38))

$$g_{ij}^{\nu} = \sum_{\mu=1}^{m} a_{ij}^{\mu\nu} \Delta_{ij}^{\mu}, \quad i, j = 1, \ldots, n, \quad \mu, \nu = 1, \ldots, m. \tag{76}$$

For i-th subsystem ($i = 2, \ldots, n$-1) the vector of resources' estimates p_i depends on its endowments of resources. The market prices $\overline{p}_1$ and $\overline{p}_n$ for corresponding markets where these resources are bought and sold are fixed. The capital dissipation can be written as

$$\sigma = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{\nu=1}^{m} g_{ij}^{\nu} \Delta_{ij}^{\nu}. \tag{77}$$

The multiplier 1/2 appears here because each flow enters (77) twice. The function σ characterises the irreversible losses necessary for maintaining resources' flows (trading costs). After taking into account (76) the capital dissipation can be rewritten as the following quadratic form

$$\sigma = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{\mu,\nu=1}^{m} a_{ij}^{\mu\nu}\Delta_{ij}^{\mu}\Delta_{ij}^{\nu} . \tag{78}$$

If all matrices A_{ij} with elements $a_{ij}^{\mu\nu}$ are positive then the matrix of this quadratic form is positive and $\sigma \geq 0$. After taking into account the reciprocity relations the condition of minimum of σ on $\bar{p}_{i\nu}$ $(i = 2, ..., n-1)$ is

$$a_{ij}^{\mu\nu} = a_{ij}^{\mu\nu}, \quad i, j = 1, ..., n; \quad \mu,\nu = 1, ..., m . \tag{79}$$

This condition and equality $\Delta_{ij}^{\nu} = p_j^{\nu} - p_i^{\nu}$ lead to the equalities

$$\sum_{j=1}^{n}\sum_{\mu=1}^{m} a_{ij}^{\mu\nu}\Delta_{ij}^{\mu} = 0, \quad i = 2, ..., n-1; \quad \nu = 1, ..., m . \tag{80}$$

which coincide with balance equations (75) if the flows have the form (76). Therefore the following statement holds: *stationary regime in an open microeconomic system that consists of internally equilibrium subsystems with flows that depend linearly on the estimates' differentials corresponds to such a distribution of resources between subsystems that capital dissipation is minimal*

This is the economic analogy of Prigogine minimal dissipation principle in irreversible thermodynamics.

CAPITAL EXTRACTION IN OPEN MICROECONOMIC SYSTEM WITH AN INTERMEDIARY

Consider economic system with an intermediary, two markets (economic reservoirs) and subsystems (Fig. 2).

The markets are described by the resources price vectors p_+ and p_-, linear resource exchange kinetics (linear dependence of flows on differentials of resources' prices (estimates))

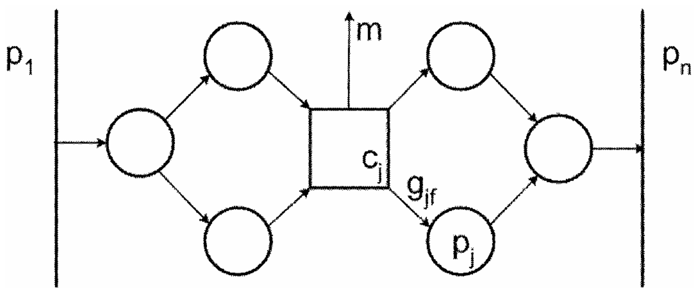

Figure 2. The structure of an open microeconomic system with an intermediary.

$$g_{ij\nu}(\Delta_{j\nu}) = \sum_{k=1}^{m} a_{ikj\nu}\Delta_{kj\nu}, \quad j,\nu = 1, ..., n . \tag{81}$$

Here $g_{ij\nu}$ is the flow of i-th resource between j and ν subsystems, $\Delta_{kj\nu} = p_{k\nu} - p_{kj}$. If one of the contacting subsystems is an intermediary which sets the price c_{kj} for buying k-th resource from i-th subsystem then $\Delta_{kjf} = c_{kf} - c_{kj}$. We denote the matrix of exchange coefficients between j-th and ν-th subsystems as $A_{j\nu}$ and between j-th subsystem and economic intermediary as A_{jf}. The flow of resource-exchange then is

$$g_{j\nu} = A_{j\nu}\Delta_{j\nu}, \quad g_{if} = A_{if}\Delta_{if} . \tag{82}$$

The flow of capital extracted from the system is

$$m = \sum_{j=1}^{n} c_j^T A_{jf} \Delta_{jf} \to \max_{c_j}, \qquad (83)$$

where c_j is the price vector with components c_{kj}. The flow of capital m attains maximum on c_j subject to the condition of non-accumulation of resources by the intermediary

$$\sum_{j=1}^{n} A_{jf} \Delta_{jf} = 0. \qquad (84)$$

The condition (84) is the system of linear equations $k = 1, ..., m$ that links the prices to the resource estimates p_{kj} for each of the passive subsystems. The problem (83) and (84) is convex and has a unique solution.

The resource estimates p_j, in turn, depend on the endowments resources N_j and capital N_{0j} as well as on the wealth function $S_j(N_j, N_{0j})$ of each economic system. They can be found from the condition that in a stationary state for any price vector $c = (c_1, ..., c_j, ..., c_n)$, the values of p_{kj} (estimates of the k-th resource in j-th subsystem) minimise the capital dissipation

$$\sigma = \frac{1}{2} \sum_{j,\nu=1}^{n} \Delta_{j\nu}^T A_{j\nu} \Delta_{j\nu} + \sum_{j=1}^{n} \Delta_{jf}^T A_{jf} \Delta_{jf} \to \min_p . \qquad (85)$$

Solution of the problems (85) and (83), (84) allows us to find the maximal flow of profit m, the corresponding resources' estimates p_j, $j = 1, ..., n$ and, if the wealth function is known, the distribution of resources between subsystems. The problem (85) should be solved subject to the condition of non-negativity of stocks N_j and N_{0j} in all subsystems that constraint the feasible set of estimates p.

CONCLUSION

In this paper we considered economic analog of the classical macro- system problem of extraction of an organized resource from a macro-system. In particular, we were concerned with the problem of extracting maximal capital from an economic system in infinite and finite times and with the problem of determining the maximal rate of capital extraction.

Conditions for the extraction of maximal capital from an open and a closed system with multi-component linear resource-exchange kinetics were obtained. The conditions that must hold for a stationary state in economic macrosystem with and without an intermediary were obtained.

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TERMODINAMIČKI MODEL IZDVAJANJA KAPITALA IZ EKONOMSKOG SUSTAVA

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SAŽETAK

U radu su proučena svojstva funkcije bogatstva ekonomskog sustava. Izveden je ekonomski analogon jednadžbe Gibbs-Duhem. Ravnotežna stanja i granični režimi izdvajanja dobiti iz neravnotežnog ekonomskog sustava su dobiveni za Cobb-Douglas funkciju bogatstva.

KLJUČNE RIJEČI

neravnotežna i ireverzibilna termodinamika, ekonomija, ekonofizika, financijska tržišta, poslovanje i menadžment