A NOVEL DISCRETE INTERNAL MODEL CONTROL METHOD FOR UNDERACTUATED SYSTEM

Imen Saidi[1, *], Islem Bejaoui[1] and Maria Gabriella Xibilia[2]

[1]University of Tunis El Manar, U.R.LARA Automatique, Ecole Nationale d'Ingénieurs de Tunis
 Tunis, Tunisia

[2]Università degli studi di Messina, Department of Engineering
 Messina, Italy

ABSTRACT

This article provides a comparative analysis of two common control configurations used to control the side-stream distillation used to separate benzene, toluene and xylene as suggested by Doukas and Lyben. Their under-actuated model has been considered as the model of distillation column and the internal model controller is designed considering a Singular Value Decomposition (SVD) and Virtual Inputs (VI) techniques. An internal controller design based on VI is proposed in this article for this kind of underactuated systems. This design is used to control in parallel the distillation process and its model in real time. The proposed controller design is simple and systematic in relation with the desired closed loop specifications of the internal model control structure. Furthermore, the controller obtained ensure robustness to process variations. The SVD technique can realize the decoupling of under-actuated processes and wipe out unrealizable factors by introducing compensation terms, affecting the dynamic of the system. The aim of this article is to make a comparison between our proposed VI controller and the SVD approach. The results we obtained confirmed the potentials of the proposed controller based on VI considering the set point tracking and its robustness.

KEY WORDS

under-actuated systems, virtual input technique, singular value decomposition technique, internal model control, stability

CLASSIFICATION

ACM: D.1.0, I.6.0, J.2
JEL: L64
PACS: 02.30.Yy, 89.20.Kk

*Corresponding author, η: imen.saidi@gmail.com; +216 97 73 4956;
 Campus Universitaire Farhat Hached el manar bp 37, le Belvedere 1002 Tunis

INTRODUCTION

The contribution of this article is study of two different control strategies for a class of chemical process industries. The model of the distillation column used in this plant, which is characterized by an underactuated structure. According to the diversity and complexity of these systems, it is important to emphasize that none of the technique proposed and developed for fully actuated systems can be applied directly to any underactuated system. Therefore, it is meaningful to develop control methods for this class of systems, and more precisely to develop the most optimized controller design.

The control objective for systems characterized by the fact that there are more degrees of freedom than actuators, is to obtain a desirable behavior of several output variables by simultaneously manipulating several inputs channels. Under-actuated systems are less sensitive to modelling errors, so it has to be controlled in its original form to obtain robust stability and performance [1].

Garcia and Morari [2], presented the concept of Internal Model Control (IMC) and have already proved the effectiveness of the framework for robust control of different kinds of Singular-Input Singular Output (SISO) systems. The IMC structure is composed of three principal parts: the process, the internal model in parallel to the process and the controller, which is the dynamic inversion of the model. Recently, many methods have been proposed to control MIMO systems with time delays. For example, Zhang and Pang [3] proposed a Closed-loop Gain Shaping Algorithm (CGSA) using Padé approximation, which is sufficiently accurate in view of stability analysis. Jin et al [4] introduced a design method of decoupling internal model control; the basic idea is to realize the decoupling of the controller of non-square processes by inserting some compensation Relative Normalized Gain Array (RNGA). An equivalent transfer function matrix is introduced to approximate the pseudo-inverse of the process transfer function matrix, which makes the design of decoupling internal model control simple and easy to calculate.

Shan and Wang [5] integrated IMC with disturbance controllers by choosing different forms of external input/output disturbance. When this disturbance is applied directly as input to the controller, the design of controller needs to compensate the effect of slow dynamic poles by adding some constraints. Pamela et al. [6] introduced an approach to regulate the heater power in those systems, which must control the temperature in food processing, pharmaceuticals and in polymerization. The objective is to control the system with both PI controller and IMC structure and to analyze its performance parameters. Jin et al. [7] proposed a novel design IMC controller based on Singular Value Decomposition (SVD), this approach uses SVD in the inverse of the steady-state gain matrix of process. The last decades have shown an increasing interest in the control of under-actuated systems, many IMC methods can be introduced to achieve considerable results on the control for these kinds of systems. For this class of system, the number of inputs is smaller than the number of outputs, which means that the transfer function matrix is not square, and the issue of inversion exists. As a result, the problem of inversion was solved in many literature researches previously mentioned [4-6] by decoupling the internal model control based on the Relative Normalized Gain Array (RNGA), Singular Value Decomposition (SVD) and an Equivalent Transfer Function (ETF) matrix. The solution brought by these methods require complicated calculations and many instructions to implement [7].

Otherwise, our proposed approach; Virtual Inputs is used to augment the system inputs inserting a certain number of virtual columns to the model in order to make it a square matrix. These virtual columns have no influence on the response of the system, and they will be eliminated

after. The control of under-actuated systems is challenging, hence the necessity to identify the control technique with less interactions and better performances such as overshoot, setting time etc. Based on internal model control for discrete under-actuated systems a comparison based on the inversion technique of the model will be made between Singular Value Decomposition (SVD) and Virtual Inputs (VI) to realize the internal model controller.

In this article, section 2 summarizes the fundamental control problem to be solved. Section 3 describe the general IMC structure. Section 4 details the design of the internal controller based on Singular Value Decomposition. Section 5 details the design of the internal controller based on Virtual Inputs. At the end, we show how our proposed methodology based on Virtual Inputs at distillation column could give good results compared to the other method described previously.

PROBLEM FORMULATION

The process to be controlled is assumed linear and discrete-time governed by the following equation [8]

$$\begin{cases} x(k+1) = Fx(k) + \sum_{i=1}^{n}\sum_{j=1}^{m} Hu(k-\tau_{ij}) \\ y(k) = Cx(k) \end{cases}$$

(1)

Where $x \in \mathbb{R}^n$ i is the system state, $u \in \mathbb{R}^m$ is the vector of manipulated variables, $y \in \mathbb{R}^n$ is the vector of system outputs, t_{ij} is the time delays, F, H and C are matrices of corresponding dimensions [8]. For the IMC design technique, it is convenient to express this model in frequency response form. Taking Z transforms of (1), the following input-output model is obtained:

$$y(z) = C(zI-F)^{-1}\sum_{i=1}^{n}\sum_{i=1}^{n} Hz^{-\tau_{ij}} u(z) = G(z)u(z)$$

(2)

Where $G(z)$ is the system transfer matrix of dimension ($n \times m$), the number of control inputs is equal to m, the number of outputs is equal to n, making it a rectangular matrix, which has the form

$$G(z) = z^{-1}\begin{pmatrix} z^{-\tau_{11}}g_{11}(z) & z^{-\tau_{12}}g_{12}(z) & \dots & z^{-\tau_{1m}}g_{1m}(z) \\ z^{-\tau_{21}}g_{21}(z) & z^{-\tau_{22}}g_{22}(z) & \dots & z^{-\tau_{2m}}g_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ z^{-\tau_{n1}}g_{n1}(z) & z^{-\tau_{n2}}g_{n2}(z) & \dots & z^{-\tau_{mn}}g_{nm}(z) \end{pmatrix}$$

(3)

Where the elements $g_{ij}(z)$, are the transfer functions of z and t_{ij} is the delay in the response of output i to input j. The synthesis of an IMC controller that is equal to the inverse of the model expression and is fundamental to ensure perfect set-point tracking and this represents the basic problem of the IMC approach. In fact, the realization of the direct model inverse is difficult, and sometimes not possible to realize, for many physical systems [8, 9]. This perfect controller cannot be implemented for the following reasons.

1.) If the model contains time delays, its inverse involves predictive terms, which make the controller unrealizable.

2.) If the zeros of the transfer matrix of the model outside the complex unit circle yield an unstable perfect controller.

3.) System equipped with a perfect controller is extremely sensitive to modelling errors and time delays.

4.) The direct model inversion is also impossible in the case of underactuated systems.

In fact, the model must provide an accurate description of the process dynamics and characteristics. Therefore, the model expression must be very close to that of the plant. For underactuated systems, the number of control inputs is less than the number of outputs and therefore we will have a rectangular matrix that is invertible. This represents the major problem encountered. In this article, to resolve this problem, it is proposed to develop some methods of inversion in the case of under-actuated systems.

INTERNAL MODEL CONTROL DESCRIPTION

The development of the IMC structure has progressed in recent years in order to design an optimal feedback controller. In this section we present the general IMC structure and we describe its basic principles and properties. Due to the fact that the traditional IMC methods cannot solve the control problem of non-square systems, we introduce two design technique to realize the internal controller but, when using a matrix to describe a non-square system, the issue of inversion often emerges.

THE GENERAL IMC DESCRIPTION

The general IMC structure of multivariable systems adopted in this article is shown in Figure 1, where $y(z)$, $y_m(z)$ and $u(z)$ are the output vector of the process, the output of internal model and the control variable, respectively. $r(z)$ is the set-point vector, $d(z)$ is the disturbance, $G(z)$ and $M(z)$ represent the transfer function matrix of process and its model, $C(z)$ is the transfer matrix of the IMC controller.
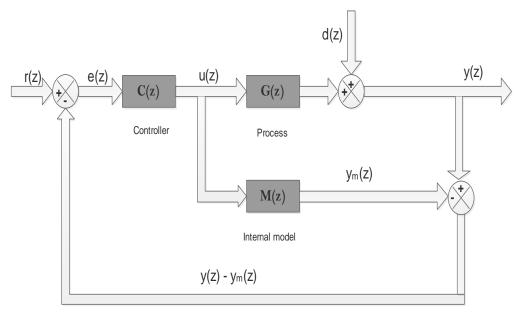

Figure 1. The general IMC structure.

We begin by reviewing the properties of the IMC structure. This structure is equivalent to a conventional feedback loop with controller. From Figure 1 the inputs vector $u(z)$ and the system outputs vector $y(z)$ are expressed by

$$u(z) = \left[I + C(z)(G(z) - M(z))\right]^{1} C(z)(r(z) - d(z)) \tag{4}$$

$$y(z) = d(z) + C(z)G(z)\left[I + C(z)(G(z) - M(z))\right]^{1}(r(z) - d(z)) \tag{5}$$

Property 1. Dual Stability; Assuming an ideal model $M(z) = G(z)$ and $d(z) = 0$, stability of both the controller $C(z)$ and the process $G(z)$ is then sufficient for overall system. Then equations (4) and (5) becomes:

$$u(z) = C(z)r(z) \tag{6}$$

$$y(z) = C(z)G(z)r(z) \tag{7}$$

Therefore, the system poles as well as the controller poles must lie inside the unit circle (UC) for stability. On the other hand, when $G(z) = M(z)$ the stability is not affected by adding constraints on the inputs.

Property 2. Perfect Control; Assume that the controller $C(z) = M(z)^{-1}$, yields a closed-loop stable IMC loop, this controller is equivalent to the inverse model to achieves perfect set-point satisfaction despite any disturbance. Furthermore, $M(z)^{-1}$ is often not realizable. When $C(z) = M(z)^{-1}$ is verified, transfer function (7) becomes

$$y(z) = M^{-1}(z)G(z)r(z) = r(z) \tag{8}$$

Property 3. Zero Offset; Assume that the steady state gain controller is equal to the inverse model gain $C(1) = M(1)^{-1}$ and the closed-loop system in Figure 1 is stable.

The key to apply the IMC structure is the controller who would yield the best output response possible. However, as in the under-actuated case, this perfect controller cannot be implemented for previous reasons. We will discuss in the next section the controller design using two different methods; Singular Value Decomposition (SVD) and Virtual Input (VI). These methods are based on the transfer function matrix of the model of the process.

IMC STRUCTURE BASED ON SINGULAR VALUE DECOMPOSITION

The application of the internal model structure to under-actuated systems is considered like our main target. In this section, we will describe the design phase of the internal controller based on SVD and the implementation at the level of the structure IMC. This approach of design can realize the decoupling of under-actuated processes and eliminate the unrealizable factors by inserting compensated terms. Meanwhile, a non-diagonal filter is designed based on SVD matrix theory. We discuss with more details this approach in the following.

The IMC structure of an under-actuated systems based on SVD is shown in Figure 2, where $C_{SVD}(z)$ is the IMC controller.
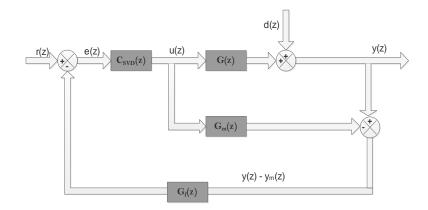
Figure 2. IMC structure based on SVD method.

The SVD of the transfer matrix $G(z)$ of the under-actuated systems is as follows:

$$G(z) = U.\Sigma.V^H$$

(9)

Where, U is an $(m' \ m)$ orthonormal matrix and $U'U = I_m$, (I_m is the identity matrix), $\sum$ is an diagonal matrix with singular values of $G(z)$ in the diagonal, V^H is an $(n \times m)$ orthonormal matrix and $V'V = I_m$ and H represented the conjugate transpose. The SVD is essentially trying to reduce the rank of the matrix and to approximate them as a linear combination [10]. The closed loop transfer function matrix deduced from Figure 2 is given by:

$$H(z) = G(z)C_{SVD}(z) \times \left[I_m + (G(z) - G_m(z)) \times C_{SVD}(z)\right]^{-1}$$

(10)

where $C_{SVD}(z)$ is a non-square internal model controller based on Singular Value Decomposition has two main roles; on one hand make the internal control, while compensating and decoupling the system to reduce coupling between channels and, on the other hand, it can satisfy the robustness and the performances of the systems. The design method of the under-actuated internal model controller $C_{SVD}(z)$ is given below.

In the nominal case, $G(z) = G_m(z)$ equation (10) becomes

$$H(z) = G(z)C_{SVD}(z) = G_m(z)C_{SVD}(z) = \text{diag}\{h_{ii}(z)\}$$

(11)

Where, $h_{ii} \neq 0$, $i = 1, 2, \ldots, m$. Hence, in the traditional IMC structure [11], the closed loop transfer function $H(z)$ and the general decoupled internal model controller $C_{IMC}(z)$ should be respectively

$$H(z) = G_m(z)C_{IMC}(z)$$

(12)

$$C_{IMC}(z) = G_m^{-1}(z)H(z)$$

(13)

It is obvious that once $G_m^{-1}(z)$ and $H(z)$ are specified, it is possible to determine $C_{IMC}(z)$. We start by finding the $G_m^{-1}(z)$ but in under-actuated systems; the exact inverse of the model does not exist, so we replace it with the generalized inverse of $G_m(z)$ [12]. Hence, the result of this inversion is the unit matrix $G_m^*(z)$:

$$G_m^*(z) = G_m^H(z)\left[G_m G_m^H\right]^{-1}$$

(14)

Where $G_m^*(z)$ and $G_m^H(z)$ are the pseudo-inverse and the Hermitian matrix of $G_m(z)$ respectively.

The expression (13) becomes

$$C_{IMC}(z) = G_m^*(z)H(z) = G_m^H(z)\left[G_m G_m^H\right]^{-1} H(z)$$

(15)

Next, we choose the appropriate $H(z)$. Whereas the model $G_m(z)$ consists into two parts to handle many limitations in this technique

$$G_m(z) = G_{m-}(z)G_{m+}(z)$$

(16)

Where $G_{m+}(z)$ contains time delays and zeros of $G_m(z)$ outside the unit circle such $G_{m-}(z)$ has a stable and realizable inverse.

In the traditional internal model controller design [12], adding the filter is used to supplement the model mismatch and ignore the error caused by the non-minimum phase portion, as shown in equation (17).

$$C_{IMC}(z) = G_m^{-1}(z)F(z)$$

(17)

Substituting equations (16) and (17) into equation (12) leads to:

$$H(z) = G(z)C_{IMC}(z) = G_m(z)C_{IMC}(z) = G_{m+}(z)G_{m-}(z)G_m^{-1}(z)F(z) = G_{m+}(z)F(z)$$

(18)

Where $F(z)$ is a designed filter of internal model controller. It can be considered as the general form [13]:

$$F(z) = \frac{b_f z^{-1}}{1 - a_f z^{-1}}$$

(19)

Among them:

$$a_f = e^{(\frac{T}{1})}$$

(20)

$$b_f = 1 - a_f$$

(21)

In the above formula, λ is the time constant of the design filter, it determines the bandwidth of the closed-loop system and thus serves as a tuning parameters for performance and robustness, T is the sampling time of the system. Furthermore, when the high performance is required, $F(z) \approx I_m$ but it is intuitively obvious that this choice makes the system very sensitive and it can very easily become unstable even for small modeling errors. In addition, the completely decoupling can be obtained by choosing $G_{m+}(z)$ and $F(z)$ diagonal. Even when dynamic interactions are allowed, $G_{m+}(z)$ and $F(z)$ must satisfy [13]:

$$G_{m+}(1) = I_m, \quad F(1) = I_m$$

(22)

In the following discussion, we give procedures for finding $G_{m+}(z)$ and a rule for filter design. Assume $G(z) = G_m(z)$, and that $G_{m+}(z)$ is diagonal with the following form [14]

$$G_{m+}(z) = diag\left\{ e^{-\tau_{ij}Tz} \prod_{P=1}^{Q_l} \left\langle \frac{-z+z_p}{-z+\bar{z}_p} \right\rangle^{ri}, i=1,2,........,m \right\}$$

(23)

where t_{ij}^T is the maximum prediction in the i-th row of $G_m^*(z)$, z_p is the pole in outside of the unit circle, $\bar{z}_p$ is the conjugate complex of z_p, r_i represents the maximum number of the same pole of $G_m^*(z)$.

IMPROVEMENT OF FILTER

According to the equation (17), the robustness of the $C_{SVD}(z)$ often cannot meet the requirement, for solving this problem, this approach need to improve the filter *F(z)*. The filtering structure is designed in general, which makes control system bear the capacity of high-dimensional decoupling and fast response. First, we use the SVD in the inverse of the transfer functions matrix of process $G(z)$. Then, we can use the term after decomposition to improve the filter $F(z)$ and to obtain the controller based on SVD.

Step 1: Use the SDV in the inverse of the steady-state gain matrix of $G(z)$

$$[G_m(z=1)]^* = U\Sigma V^H$$

(24)

Step 2: Let W_v satisfy the following formula

$$V^H W_V = I$$

(25)

Step 3: The improved internal model controller is as follows

$$C_{SDV}(z) = G_m^*(z)G_{m+}(z)W_v F(z)W_v^{-1}$$

(26)

The robustness of the system can be greatly enhanced by adding a filter $G_f(z)$ in the feedback loop, and the filter time constant can be set to half of the maximum delay time in this loop [15].

IMC STRUCTURE BASED ON VIRTUAL INPUTS (VI)

Focusing now on the inversion method of the under-actuated systems based on Virtual Inputs. To successfully apply our approach, firstly we need to modify the IMC basic structure mentioned in Figure 1, so that it becomes applicable to underactuated systems with more outputs than control inputs. Secondly, we design an approximate inverse of the model plant which is inspired by the studies of [16, 17] in the case of MIMO systems and over-actuated systems [18].

The modified IMC structure presented in Figure 3 is characterized by two more blocks with respect to the basic IMC structure presented in Figure 1 [19]. The first new block is the Virtual Inputs Augmentation VIA (z) which is used to augment the system inputs inserting virtual ($n \times (n - m)$) column to the transfer matrix of the non-square system in order to make it of dimension ($n \times m$), so that it can be inverted. The second one is the Virtual Inputs Removing VIR (z) block which is used to eliminate the exceeding virtual inputs [19].
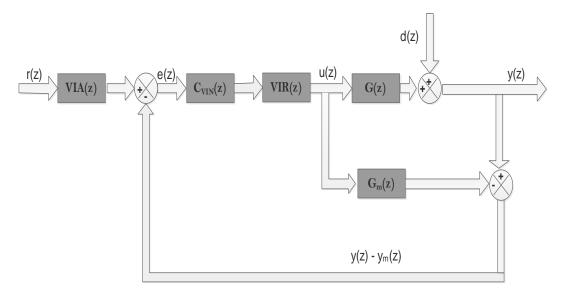
Figure 3. IMC structure based on Virtual Inputs method.

The inserting columns mentioned previously can be chosen as first-order transfer functions, which verify the stability criterion, and in order to simplify the study and avoid inversion problems [19].

The studied system can be shown through the following equation [18, 19]:

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} G_{11}(z) & G_{12}(z) & \cdots & G_{1m}(z) \\ G_{21}(z) & G_{22}(z) & \ldots & G_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1}(z) & G_{n2}(z) & \ldots & G_{nm}(z) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}
$$

(27)

When the model and the system match perfectly $G(z) = G_m(z)$, the augmented system $\breve{G}(z)$ and its exact model are expressed as:

$$
\tilde{G}(z) = \tilde{G}_m(z) = \underbrace{\begin{bmatrix} G_{11} & \ldots & G_{1m} \\ \vdots & \ddots & \vdots \\ G_{n1} & \ldots & G_{nm} \end{bmatrix}}_{\substack{\text{Initial transfer function} \\ (n \times m)}} \underbrace{\begin{bmatrix} G_{1m+1} & \ldots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{nm+1} & \ldots & G_{nn} \end{bmatrix}}_{\substack{\text{Added transfer function} \\ (n \times (n-m))}}
$$

(28)

The Virtual Inputs Controller C_{VIN} *(z)* design presented in Figure 4 is based on the inversion method reported in [18, 19].

The closed-loop transfer function matrix C_{VIN} (z) between *e(z)* and *u(z)* is derived as:

$$
C_{VIN}(z) = K_2 (I_n + K_1 \tilde{G}_m(z))^{-1} K_1
$$

(29)

Where K_1 is an invertible square matrix $(n \times n)$ used to ensure the stability of the controller and it can be expressed by $K_1 = \alpha I_m, \alpha \in R^+$, a second gain K_2 is introduced as reported in Figure 4 in order to compensate the static errors. The gain K_2 is given by equation (30).

$$
K_2 = K_1^{-1} (I_m + K_1 \tilde{G}_m(1))^{-1} \tilde{G}_m(1)^{-1}
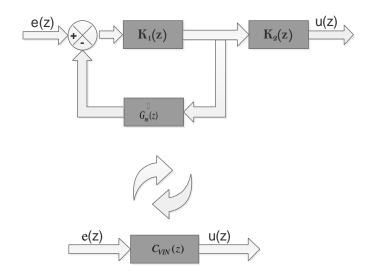$$

(30)

Figure 4. Structure of the Virtual Input controller.

For sufficiently high values of a , the controller $C_{VIN}(z)$ approaches the inverse of internal model $\check{G}_m(z)$:

$$C_{VIN}(z) \approx \tilde{G}_m(z)^{-1}$$

(31)

For the construction of the controller, it is necessary to respect the fact that:

$$C_{VIN}(z).\tilde{G}_m(z) = I_n$$

(32)

The virtual inputs method proposed for the design of the controller by internal model is valid whatever the number of outputs $n \in \mathbb{N}^*$ and inputs $m \in \mathbb{N}^*$ of a physical system. In fact, this approach is applicable for the following classes of systems:

- Monovariable system where $n = m = 1$,
- Overactuated system where $n < m$,
- Underactuated system where $n > m$.

SIMULATION RESULTS AND COMPARISON

To analyze the comparative study on the above control technique, we considered a chemical process industry. In order to test the control effect of discrete underactuated internal model controller based on SVD and VI, a side-stream distillation control problem suggested by Doukas and Lyben will be used [20].

Figure 5 shows the side-stream distillation scheme that serves to separate benzene, toluene and xylene. The problem posed in this case was to control the concentrations of four impurities in three product streams with only three manipulated variables: reboiler duty, reflux ratio and side stream flow rate.

Doukas and Lyben [20] treated the distillation column as a 4×3 system. An internal controller should be design for the under-actuated system. Doukas and Lyben model can be represented as

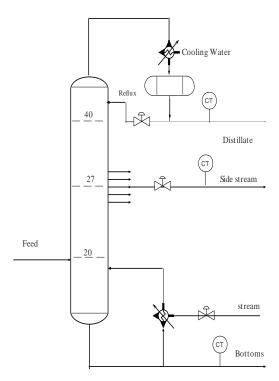**Figure 5.** Schematic diagram of distillation column [20].

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} G_{11}(z) & G_{12}(z) & G_{13}(z) \\ G_{21}(z) & G_{22}(z) & G_{23}(z) \\ G_{31}(z) & G_{32}(z) & G_{33}(z) \\ G_{41}(z) & G_{42}(z) & G_{43}(z) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{33}$$

where the toluene in bottom (y_1), toluene in bottom (y_2), benzene in side draw (y_3) and benzene in side draw (y_4) are the four controlled variables. The reboiler duty (u_1), reflux ratio (u_2) and side draw (u_3) are the manipulated variables [20].

The $(4' 3)$ transfer function matrix presented in equation (33) as follows

$$G_{11}(z) = \frac{-5.12\ z - 0.61}{z - 0.41} z^{-1}, G_{12}(z) = \frac{0.0017\ z^2 + 0.0356\ z + 0.0117}{z^2 - 1.27\ z + 0.40} z^{-1} \tag{34}$$

$$G_{13}(z) = \frac{-0.3811\ z^2 - 1.078\ z - 0.07728}{z^2 - 1.263\ z + 0.3985} z^{-1}, G_{21}(z) = \frac{2.507z + 0.5044}{z - 0.4967} z^{-1} \tag{35}$$

$$G_{22} = \frac{-0.0175\ z^2 - 0.0208\ z - 0.0566}{z^2 - 1.72\ z + 0.74} z^{-1}, G_{23} = \frac{0.0016\ z + 0.0015}{z^2 - 1.95\ z + 0.95} z^{-6} \tag{36}$$

$$G_{31}(z) = \frac{2.35\ z^2 + 0.0182\ z + 0.17}{z^2 - 0.0018\ z + 0.0077} z^{-1}, G_{32}(z) = \frac{0.0077\ z^2 + 0.003837\ z + 0.0111}{z^2 - 0.49\ z + 0.0607} z^{-1} \tag{37}$$

$$G_{33} = \frac{-0.29\ z^2 - 0.0225\ z - 0.0091}{z^2 - 0.0299\ z + 0.0002} z^{-1}, G_{41} = \frac{-5.66\ z - 0.87}{z - 0.44} z^{-1} \tag{38}$$

$$G_{42} = \frac{-0.0706\ z^2 - 0.0324\ z - 0.0166}{z^2 - 0.46\ z + 0.0551} z^{-1}, G_{43} = \frac{0.94\ z^2 + 0.63\ z + 0.0008}{z^2 - 0.81\ z + 0.16} z^{-1} \tag{39}$$

In this section, we evaluate the controller performance using the above-mentioned approach.

SIMULATION RESULTS USING THE SVD CONTROLLER

Starting with the internal controller based on Singular Value Decomposition, and using the whole approach seen in section 4. Using equation (23), $G_{m+}(z)$ can be obtained by

$$G_{m+}(z) = diag\left\{\frac{1}{z-7.75}, \ \frac{1}{z-60}, \ \frac{1}{z-0.59}, \ \frac{1}{z-1.91}\right\}$$

(40)

Using equation (24), L, S, V is given as follows

$$U = \begin{bmatrix} -0.4523 & -0.4230 & 0.7841 & 0.0399 \\ -0.8883 & 0.1387 & -0.4377 & 0.0022 \\ 0.0327 & -0.1356 & -0.1043 & 0.9847 \\ -0.0720 & 0.8851 & 0.4274 & 0.1696 \end{bmatrix}$$

(41)

$$\Sigma = \begin{bmatrix} 0.6198 & 0 & 0 \\ 0 & 0.0902 & 0 \\ 0 & 0 & 0.0553 \end{bmatrix}$$

(42)

$$V = \begin{bmatrix} 0.0243 & -0.5117 & -0.8588 \\ 0.9968 & -0.0534 & 0.0600 \\ 0.0765 & 0.8575 & -0.5088 \end{bmatrix}$$

(43)

Using equation (25), W_v, W_v^{-1} are given as follows

$$W_v = \begin{bmatrix} 0.0243 & 0.9968 & 0.0765 \\ -0.5117 & -0.0534 & 0.8575 \\ -0.8588 & 0.0600 & -0.5088 \end{bmatrix}$$

(44)

$$W_v^{-1} = \begin{bmatrix} 0.0243 & -0.5117 & -0.8588 \\ 0.9967 & -0.533 & 0.0600 \\ 0.0766 & 0.8575 & -0.5087 \end{bmatrix}$$

(45)

Considering the realization of the controller, the added filter $F(z)$ is

$$F(z) = \begin{bmatrix} \dfrac{0.0007\,z + 0.0007}{z^2 - 1.99\,z + 0.99} & \dfrac{0.0001\,z + 0.0001}{z^2 - 1.96\,z + 0.96} & \dfrac{0.0007\,z + 0.0007}{z^2 - 1.92\,z + 0.92} & \dfrac{0.0003\,z + 0.0003}{z^2 - 1.94\,z + 0.94} \\ \dfrac{0.0007\,z + 0.0007}{z^2 - 1.99\,z + 0.992} & \dfrac{0.0001\,z + 0.0001}{z^2 - 1.96\,z + 0.96} & \dfrac{0.0007\,z + 0.0007}{z^2 - 1.92\,z + 0.92} & \dfrac{0.0003\,z + 0.0003}{z^2 - 1.94\,z + 0.94} \\ \dfrac{0.0007\,z + 0.0007}{z^2 - 1.99\,z + 0.99} & \dfrac{0.0001\,z + 0.0001}{z^2 - 1.96\,z + 0.96} & \dfrac{0.0007\,z + 0.0007}{z^2 - 1.92\,z + 0.92} & \dfrac{0.0003\,z + 0.0003}{z^2 - 1.94\,z + 0.94} \end{bmatrix}$$

(46)

In order to increase the robustness of system, we add a feedback filter in feedback loop. The expression of the added filter is as follow:

$$G(z) = diag\left\{G_{f11}(z) \quad G_{f22}(z) \quad G_{f33}(z) \quad G_{f44}(z)\right\}$$

(47)

where,

$$G_{f11}(z) = \frac{z}{z^2 - 0.001z + 0.004}, G_{f22}(z) = \frac{z - 0.0001422}{z^2 - 0.0002846z + 0.00000125} \quad (48)$$

$$G_{f33}(z) = \frac{0.9971z - 0.006362}{z^2 - 0.009359z + 0.0001379}, G_{f44}(z) = \frac{z - 0.00008516}{z^2 - 0.000001784z} \quad (49)$$

The expression of the inverse of $G_m(z)$, $G_m^*(z)$ is given as follows

$$G_m^*(z) = \begin{bmatrix} G_{m11}^*(z) & G_{m12}^*(z) & G_{m13}^*(z) & G_{m14}^*(z) \\ G_{m21}^*(z) & G_{m22}^*(z) & G_{m23}^*(z) & G_{m24}^*(z) \\ G_{m31}^*(z) & G_{m32}^*(z) & G_{m33}^*(z) & G_{m34}^*(z) \end{bmatrix}$$

(50)

where,

$$G_{m11}^* = \frac{-0.023z^2 - 0.0006079z}{z^2 - 0.0366z + 0.0003355} z^{-1}, G_{m12}^* = \frac{0.000699z - 0.0005}{z^2 - 0.2707z + 0.01832} \quad (51)$$

$$G_{m13}^* = \frac{0.01035z^2 + 0.000875z}{z^2 - 0.03663z + 0.0003355} z^{-1}, G_{m14}^* = \frac{-0.04769z^2 - 0.006181z}{z^2 - 0.139z + 0.004828} z^{-1} \quad (52)$$

$$G_{m21}^* = \frac{-0.3386z^2 + 0.07526z - 0.001504}{z^2 - 0.03663z + 0.0003355}, G_{m22}^* = \frac{-0.7011z^2 + 0.3355z - 0.04626}{z^2 - 0.2707z + 0.01832} z^{-1} \quad (53)$$

$$G_{m23}^* = \frac{0.01847z^2 + 0.001332z}{z^2 - 0.03663z + 0.0003355} z^{-1}, G_{m24}^* = \frac{-0.0025z^2 - 0.03611z - 0.00231}{z^2 - 0.139z + 0.004828} \quad (54)$$

$$G_{m31}^* = \frac{-0.07642z^2 + 0.003053z}{z^2 - 0.03663z + 0.0003355} z^{-1}, G_{m32}^* = \frac{0.105z^2 - 0.05384z - 0.06541}{z^2 - 0.2707z + 0.01832} z^{-1} \quad (55)$$

$$G_{m33}^* = \frac{-1.182e\text{-}08z^2 - 7.598e\text{-}07z}{z^2 - 0.03663z + 0.0003355}, G_{m34}^* = \frac{0.04368z^2 + 0.002231z - 7.429e\text{-}06}{z^2 - 0.139z + 0.004828} \quad (56)$$

The final expression of the internal controller based on SVD described by the equation (26) is the following

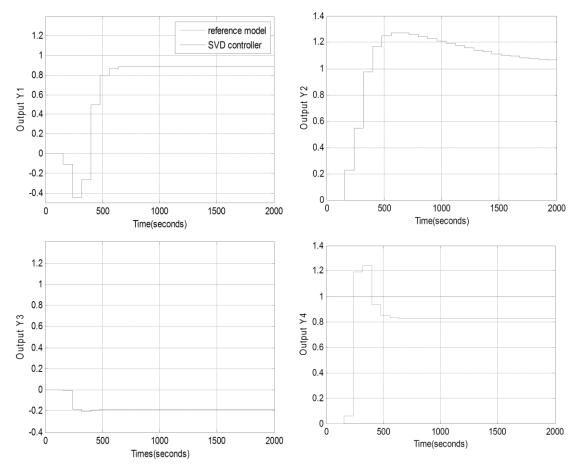$$C_{SDV}(z) = G_m^*(z) G_{m+}(z) W_v F(z) W_v^{-1} \quad (57)$$

Figure 6. Outputs System with internal controller based on SVD.

SIMULATION RESULTS USING THE VIRTUAL INPUT CONTROLLER

Dealing now with our proposed approach, the Virtual Input methods applied on the same system studied previously. Considering the same underactuated system with three control and four outputs. The system transfer matrix $G(z)$ is given by (33).

Let us consider the case of perfect modeling $G(z) = G_m(z)$, The augmented model transfer function is of dimension 4×4. The system and the model outputs are expressed respectively by (58) and (59).

$$[y_1 \quad y_2 \quad y_3 \quad y_4]^T = G(z)u \tag{58}$$

$$[y_{m1} \quad y_{m2} \quad y_{m3} \quad y_{m4}]^T = \begin{bmatrix} G_{m11} & G_{m12} & G_{m13} & G_{m14} \\ G_{m21} & G_{m22} & G_{m23} & G_{m24} \\ G_{m31} & G_{m32} & G_{m33} & G_{m34} \\ G_{m41} & G_{m42} & G_{m43} & G_{m44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{59}$$

The model transfer functions G_{m11} until G_{m43} are successively chosen to be close to G_{11} until G_{43}. The augmenting $(n \times (n-m))$ virtual column $G_{m14}, G_{m24}, G_{m34}$ and G_{m44} are chosen as first order transfer functions so that they ensure the invertibility conditions of the matrix $G_m(z)$.

It is mandatory to study the discrete-time internal controller stability, in order to set the stability interval of the gain matrix K_1 which ensures the stability of the IMC structure. The expression of the controller (29) detailed in the previous section can be reformulated using the representation of sate as:

$$\begin{cases} x(k+1) = F\,x(k) + H\,e(k) \\ u(k) = C\,x(k) \end{cases}$$

(60)

Where $x(k) \in \mathbb{R}^n$, $e(k) \in \mathbb{R}^n$ and $u(k)\ \mathbb{R}^n$, are the state, input and output vectors respectively of the controller, the matrices F, H and C are known constant matrices. The stability condition highlighted by Lyapunov theory allows to assess the necessary and sufficient condition of the controller stability. Using this theory, we can conclude that the system presented by the equation (60) is stable if and only if there exists a positive definite matrix $P = P^T > 0$, satisfying the following Lyapunov inequality [21]:

$$P > 0\,,\ \left(F^T P F - P\right) < 0$$

(61)

Solving the LMI equation (61), the interval of the gain K_1 which assures the stability of the internal controller is $-\,0.1 < K_1 < 0.01 \times I_4$. In the case of this system, we choose $a = 0.01$ such that $K_1 = \alpha \times I_4$.

The gain matrix K_2 relative to $K_1 = 0.01 \times I_4$ is given by:

$$K_2 = \begin{bmatrix} 1.1869 & -0.0177 & -0.7142 & 0.5450 \\ 0.9388 & 5.3832 & -5.6314 & 0.3094 \\ 0.6865 & 0.1494 & 0.7212 & -0.5571 \\ 0.0733 & 0.0105 & 0.8240 & 1.1118 \end{bmatrix}$$

(62)

The set responses of the studied system with both controller design is presented through the Figure 6 and Figure 7; we note that the steady state error is not null in the case where we used the controller based on SVD.

COMPARISON ANALYSIS

After using both Singular Value Decomposition and Virtual Inputs approaches, it seems to be quite interesting making a comparison between them showing their effectiveness in terms of stability, robustness, precision and tracking signal.

From the data presented in table 1, it is relevant to note that the desired specifications of the closed-loop responses are not met with the SDV controller and this is expected because of the problem of the interaction which are too strange as illustrated by the decoupling of the controller and the simulation results obtained with the SVD controller in Figure 6. SVD method has a problem if we do not make the right choice of the parameter of the filter, in this case, the approximation cannot be made with effectiveness and the system can be diverging. Added to that this approach towards the under-actuated process is to square the system by make the decoupling of the model affecting the characteristic of the system. Unfortunately, this operation can decrease the performance of the system and making it so poor by neglecting some information's.
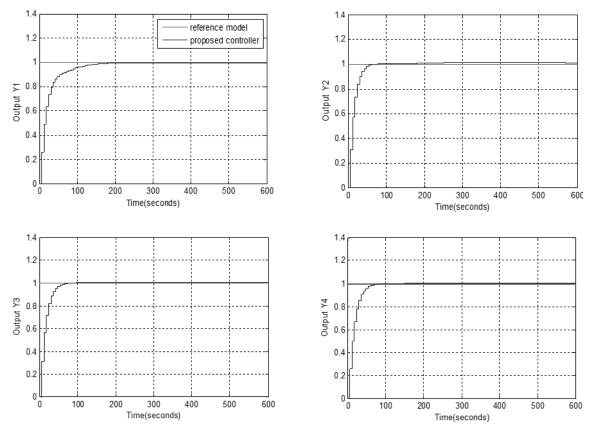
Figure 7. System outputs with internal controller based on Inputs Virtual.

The internal model controller using a Virtual Inputs as inversion approach of the model $G_m(z)$ better performances in terms of rise time, settling time and in terms of error dynamics precision are obtained, as shown by table1, these results are satisfactory because the steady state is zero for all responses as shown in Figure 7. This mean that the set-point tracking is ensured. The internal model control using the Virtual Input approach is having the benefits of small overshoot, faster tracking features, and less interaction compared to the Singular Value Decomposition method as indicated by the Figures 6 and 7 and by the table below.

The Virtual Input method may be fails when the system has a disturbance acting on the outputs of the system. For this reason, with load disturbance the response needs to be controlled to attain robustness and performance.

Table 1. Quantitative comparison between IMC Controller based SVD and Virtual Inputs.

		Outputs of System			
Parameters to control		**Y1**	**Y2**	**Y3**	**Y4**
IMC controller based on SVD	Rise Time (s)	93	160,6	-6	240
	Steady state error (%)	12,99	1,96	66,24	21,36
	Settling time (s)	560	320	247	400
IMC controller based on VI	Rise Time (s)	53,95	23,92	30,02	30,14
	Steady state error (%)	0	0	0	0
	Settling time (s)	96,02	42,05	42,05	47,97

In order to verify the robustness of the proposed internal model controller, we added a white noise with a variation of 0,2 acting on the outputs of the system in the interval of time 100 s, 150 s, 200 s, 250 s. The influence of this noise is observed in Figure 8.
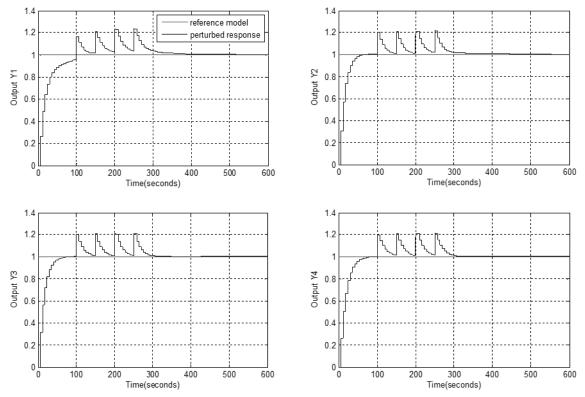

Figure 8. System outputs with disturbed internal controller based on Virtual Inputs.

It is shown that the closed loop system is not sensitive to noise. We conclude that the method it offers strong robustness in case of the given perturbation and good tracking features.

CONCLUSION

To control an under-actuated industrial process demands not only to know all the characteristic of his model, but also to achieve the desired performance when there exist load disturbances. Both Virtual Input and SVD approaches are applied to distillation column process, to prove the controller achievability for this industrial process and that it is the most efficient approach in terms of rapid response, set-point tracking and disturbance rejection. Through a comparative analysis, the Virtual Inputs approach avoids the complex calculation, such as calculate the inverse of the matrix, the controller structure is simple, has a few tuning parameters, and is easy to be accepted by operators.

The simulation results show that our proposed method (Virtual Inputs) ensures suitably the set-point tracking and the disturbance rejection for the external disturbance as illustrated in distillation control problem suggested by Doukas and Lyben and gives better results compared to SVD approach. The modified internal model control scheme gives more degree of freedom in the controller design technique in order to improve the performance of the controlled system.

As future works, it will be interesting to handle with unstable multivariable non-square systems and non-linear systems as proposed in order to design an internal controller with more flexibility and to apply the proposed Virtual Input controller to real processes.

REFERENCES

[1] De Luca, A.; Mattone, R. and Oriolo, G.: *Stabilization of underactuated planar 2R manipulator*.
International Journal of Robust and Nonlinear Control **10**(4), 181-198, 2000,
http://dx.doi.org/10.1002/(SICI)1099-1239(20000415)10:4<181::AID-RNC471>3.0.CO;2-X,

[2] Garcia, C.E. and Morari, M.: *Internal model Control. 2. Design procedure for multivariable systems*.
Industrial & Engineering Chemistry Process Design and Development **24**(2), 472-484, 1985,
http://dx.doi.org/10.1021/i200029a043,

[3] Zhang, X. and Pang, H.: *Novel concise robust control design for non-square systems with multiple time delays*.
Journal of Nature and Science **1**(2), 1-4, 2015,

[4] Jin, Q.; Jiang, B.; Wang, Q. and Shan, G.: *Decoupling internal model control for non-square processes based on equivalent transfer function*.
Transaction of the Institute of Measurment and Control **36**(8), 1114-1131, 2014,
http://dx.doi.org/10.1177/0142331214534110,

[5] Shan, G. and Wang Q.: *The design of disturbances rejection controller for decoupling internal model control system of non-square processes*.
Proceedings of the International Workshop on Computer Science in Sports, 2013,
http://dx.doi.org/10.2991/iwcss-13.2013.41,

[6] Pamela, D.; Jebrajan, T. and Baby, B.: *Real time implementation of internal model controller for temperature process*.
International Conference on Renewable Energy and Sustainable Energy. Coimbatore, pp.162-165, 2013,
http://dx.doi.org/10.1109/ICRESE.2013.6927807,

[7] Bing-Jin, Q.; Zhao, L.; Hao, F. and Wen Liu, S.: *Design of a multivariable internal model controller based on singular value decomposition*.
The Canadian Journal of Chemical Engineering **91**(6), 1103-1114, 2013,
http://dx.doi.org/10.1002/cjce.21735,

[8] Morari, M. and Garcia, C.E.: *Internal model control, A unifying review and some new results*.
Industrial & Engineering Chemistry Process Design and Development **21**(2), 308-323, 1982,
http://dx.doi.org/10.1021/i200017a016,

[9] Touati, N.; Saidi, I.; Dhahri, A. and Soudani, D.: *Internal multimodel control for nonlinear overactuated systems*.
Arabian Journal for Science and Engineering **44**(3), 2369-2377, 2019,
http://dx.doi.org/10.1007/s13369-018-3515-5,

[10] Qi-Bing, J.; Si-Wen, L.; Ling, Q. and Li-Ting, C.: *Internal model control based on singular value decompostion and its application to non-square processes*.
Acta Automatica Sinica **37**(3), 354-359, 2011,
http://dx.doi.org/10.3724/SP.J.1004.2011.00354,

[11] Tian, C.: *Decoupling internal model control method for operation of indutrial process*.
Journal of ACTA Automatica Sinica **35**(10), 1362-1368, 2009,
http://dx.doi.org/10.3724/sp.j.1004.2009.01362,

[12] Penrose, R.: *A generalized inverse for matrices*.
Proceedings of the Cambridge Philosophical Society **51**(3), 406-413, 1955,
http://dx.doi.org/10.1017/S0305004100030401,

[13] Sit Lee, W. and Anderson, B.D.O.: *New filters for internal model control design*.
International Journal of Robust and Nonlinear Control **4**(6), 757-775, 1994,
http://dx.doi.org/10.1002/rnc.4590040605,

[14] Wang, Q.G.; Zhang, Y. and Chiu, M.S.: *Decoupling internal model control for multivariable systems with multiple time delays*.
Chemical Engineering Science **57**(1), 115-124, 2002,
http://dx.doi.org/10.1016/S0009-2509(01)00365-7,

[15] Normey-Rico, J.E.; Bordons, C. and Camacho, E.F.: *Improving the robustness of dead-time compensationg PI controllers*.
Control Engineering Practice **5**(6), 801-810, 1997,
http://dx.doi.org/10.1016/S0967-0661(97)00064-6,

[16] Chen, J.; Zhang, B. and Qi, X.: *A new control method for MIMO first order time delay non-square system*.
Journal of Process Control **21**(4), 538-546, 2011,
http://dx.doi.org/10.1016/j.jprocont.2011.01.007,

[17] Bejaoui, I.; Saidi, I. and Soudani, D.: *Internal model control of MIMO discrete under-actuated systems with real parametric uncertainty*.
Proceedings of the 2nd International Conference on Advanced Systems and electric Technologie, Hammamet, pp.308-314, 2018,
http://dx.doi.org/10.1109/ASET.2018.8379874,

[18] Saidi, I.; Touati, N.; Dhahri, A. and Soudani, D.: *A comparative study on existing and new methods to design internal model controllers for non-square systems*.
Transactions of the Institute of Measurement and Control **41**(13), 3637-3650, 2019,
http://dx.doi.org/10.1177/0142331219834608,

[19] Bejaoui, I.; Saidi, I. and Soudani, D.: *Internal model control of MIMO discrete under-actuated systems with real parametric uncertainty*.
Proceedings of the 2nd International Conference on Advanced Systems and electric Technologie, Hammamet, pp.308-314, 2018,
http://dx.doi.org/10.1109/ASET.2018.8379874,

[20] Bhat-Vinayambika, S.; Shanmuga-Priya, S. and Thirunavukkarasu I.: *A Comparative Study on Control Techniques of Non-square Matrix Distillation Column*.
International Journal of Control Theory and Applications **8**(3), 1129-1136, 2015,

[21] Chang, J.W. and Yu, C.C.: *The relative gain for non-square multivariable systems*.
Chemical Enginnering Science **45**(5), 1309-1323, 1990,
http://dx,doi.org/10.1016/0009-2509(90)87123-A.