

MATHEMATICAL FORMALISATION OF EMERGENT BEHAVIOUR IN DIGITAL TWINS OF MANUFACTURING SYSTEMS

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ABSTRACT

The rapid development of digital twin technology has created new opportunities for modelling and optimisation of complex manufacturing systems. However, most existing studies treat digital twins primarily as simulation tools and seldom address emergent behaviour arising from subsystem interactions. Emergence is a fundamental property of complex systems in which global behaviour cannot be reduced to the sum of individual component behaviours. This article proposes a mathematical framework for analysing emergent behaviour in digital twins of manufacturing systems. The digital twin is modelled as a dynamic network of interacting subsystems described by a system of differential equations. An entropy-based emergence index is introduced to quantify collective complexity. Two stability theorems and one optimality theorem are formally proved. A variational Principle of Minimum Emergent Complexity is formulated, linking information-theoretic complexity directly to manufacturing efficiency. Numerical simulations with four scientific figures confirm the theoretical predictions. The proposed formalisation provides a foundation for intelligent control in Industry 5.0 environments.

KEY WORDS

digital twin, complex systems, emergence, manufacturing systems, entropy

CLASSIFICATION

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INTRODUCTION

The concept of a digital twin has become one of the key technological paradigms of Industry 4.0 and Industry 5.0 [1, 2]. A digital twin is a virtual representation of a physical system that enables real-time monitoring, simulation, and optimisation of technological processes.

Manufacturing systems are inherently complex and consist of multiple interacting subsystems – machine tools, robotic manipulators, transport modules, and control systems. In such systems global behaviour often emerges from component interactions rather than from the behaviour of individual elements. This phenomenon is known as emergent behaviour and is characteristic of complex adaptive systems [3, 4].

Despite growing interest in digital twin technology, the mathematical description of emergent behaviour in digital twins of manufacturing systems remains insufficiently explored. Most existing studies focus on real-time monitoring [1], predictive maintenance [2], and process optimisation [5], treating digital twins mainly as simulation tools. Manufacturing environments, however, represent complex adaptive systems characterised by nonlinear interactions whose collective dynamics exhibit properties that cannot be inferred from local component analysis alone [6].

The aim of this article is to bridge this gap by developing a rigorous mathematical framework for describing and quantifying emergent behaviour in digital twins of manufacturing systems, proving stability conditions for emergent dynamics, and establishing a variational optimality principle that links emergent complexity to manufacturing efficiency.

RELATED WORK AND SCIENTIFIC CONTEXT

Early conceptual work by Grieves and Vickers [1] defined the digital twin as a dynamic virtual model capable of reflecting the physical state of an object throughout its lifecycle. Subsequent studies by Tao and Zhang [2] expanded this concept to intelligent manufacturing systems integrating data analytics and artificial intelligence. Kritzinger et al. [5] provided a systematic review differentiating digital models, digital shadows, and digital twins.

The phenomenon of emergence has been studied extensively in complex systems theory. Bar-Yam [3] characterised emergent behaviour as the appearance of system-level properties irreducible to individual element properties. Shannon's information-theoretic framework [7] established entropy as a rigorous measure of system uncertainty and complexity. Holland [8] formalised emergence in complex adaptive systems, while Prigogine and Stengers [9] demonstrated self-organisation phenomena in nonequilibrium physical systems.

At the intersection of these fields, the mathematical description of emergent dynamics in digital twin environments remains largely unexplored. The present article attempts to fill this gap by introducing a quantitative emergence index applicable to cyber-physical manufacturing systems.

DIGITAL TWIN AS A COMPLEX DYNAMIC SYSTEM

Consider a manufacturing system consisting of N interacting subsystems. The state of the system is described by the vector:

$$X(t) = \{x_1(t), x_2(t), \dots, x_n(t)\}, \quad (1)$$

where $x_i(t)$ are the state variables of the i -th subsystem. In isolation, the dynamics of each subsystem is:

$$dx_i/dt = f_i(x_i, u_i, t), \quad (2)$$

where u_i denotes control parameters. In a real manufacturing system, subsystems interact. Incorporating coupling, the full system dynamics become:

$$dx_i/dt = f_i(x_i, u_i, t) + \sum_{j=1..N} a_{ij} g(x_i, x_j), \quad (3)$$

where $A = [a_{ij}]$ is the interaction matrix defining the topology of the cyber-physical manufacturing system, and $g(x_i, x_j)$ is the coupling function between subsystems.

ENTROPY-BASED EMERGENCE INDEX

To quantify the complexity of system behaviour, define the normalised probability distribution of subsystem states:

$$p_i(t) = x_i(t) / \sum_k x_k(t). \quad (4)$$

The Shannon informational entropy of the system is:

$$H(t) = -\sum_i p_i(t) \cdot \log p_i(t). \quad (5)$$

Entropy reflects the level of disorder or uncertainty in the distribution of system activity. The emergence index is defined as:

$$E(t) = dH(t)/dt, \quad (6)$$

where $E(t) = 0$ is stationary regime; $E(t) > 0$ growing complexity; while $E(t) < 0$ represents self-organisation and convergence to a collective regime.

ANALYTICAL PROPERTIES OF THE EMERGENCE INDEX

Because $p_i = 1$ implies $\sum (dp_i/dt) = 0$, differentiation yields:

$$E(t) = -\sum_i (dp_i/dt) \cdot \log p_i. \quad (7)$$

Property 1 (Non-negativity at uniform distribution): when $p_i \rightarrow 1/N$, entropy attains $H_{\max} = \log N$ and $E(t) = 0$.

Property 2 (Sensitivity to interaction intensity): let $a_{ij} = kb_{ij}$. For small k , $E(t) \approx 0$; for large k , $E(t) > 0$, indicating strong collective dynamics.

Property 3 (Threshold of collective behaviour): collective behaviour emerges when the spectral radius $\rho(A)$ exceeds the dissipation coefficient α .

STABILITY ANALYSIS OF EMERGENT DYNAMICS

Consider the linear interaction model:

$$dX/dt = (A - \alpha I) X, \quad (8)$$

$B = A - \alpha I$ are $\mu_i = \lambda_i - \alpha$, where λ_i are eigenvalues of A .

Theorem 1 (Stabilisation Condition): The digital twin system is asymptotically stable if and only if $\lambda_{\max}^x < \alpha$, and unstable if $\lambda_{\max}^x > \alpha$.

Proof. The general solution is $X(t) = e^{(Bt)} X(0)$. Asymptotic stability requires $\text{Re}(\mu_i) < 0$ for all i , i.e., $\lambda_i < \alpha$. For the dominant eigenvalue: $\lambda_{\max}^x < \alpha$. $\square$

Theorem 2 (Emergence Condition): Emergent collective behaviour arises if and only if $\rho(A) > \alpha$.

Proof. When $\rho(A) > \alpha$, there exists $\lambda_i > \alpha$, so $\text{Re}(\mu_i) > 0$. The corresponding mode grows exponentially, inducing collective dynamics that dominate local dissipation. $\square$

These two theorems establish a sharp spectral criterion for the onset of emergent behaviour in the cyber-physical manufacturing system.

CRITICAL EMERGENCE AND PHASE STRUCTURE

The system exhibits a phase transition at the critical interaction parameter $k^c = \alpha/\rho(\hat{B})$. Three operational phases are identified:

- Phase I – Stable operation: $\rho(A) < \alpha$. Subsystems operate independently; $E(t) \rightarrow 0$,
- Phase II – Critical zone: $\rho(A) \approx \alpha$. System is maximally sensitive to small perturbations,
- Phase III – Emergent dynamics: $\rho(A) > \alpha$. Collective behaviour dominates; subsystem states synchronise.

Definition (Critical Emergence). The digital twin exhibits ‘critical emergence’ when $\rho(A) = \alpha$. In manufacturing, this manifests as bottleneck formation, cascade load accumulation, or machine synchronisation.

A normalised emergence index $E_n(t) = E(t)/H_{\max} \in [-1, 1]$ is introduced. The transition criterion is $|E_n(t)| > E_{n, \text{crit}}$, where $E_{n, \text{crit}}$ is a system-specific threshold.

NONLINEAR EXTENSION: CATASTROPHE-LIKE BIFURCATION

The linear model of Section 3 may be extended to include nonlinear saturation effects:

$$dx_i/dt = -\alpha x_i + \sum_j a_{ij} x_j - \gamma x_i^3, \quad (9)$$

where $\gamma > 0$ is a nonlinear dissipation coefficient. This cubic term introduces an S-shaped (catastrophe-like) bifurcation in the steady-state emergence index as a function of $\rho(A)$. The S-curve reveals three regime branches:

- lower stable branch ($\rho(A) \ll \alpha$) – subsystems operate as independent modules; $E_\infty \approx 0$,
- unstable intermediate branch ($\rho_1 < \rho(A) < \rho_2$) – the system may jump between stable and emergent regimes; this is the zone of structural instability,
- upper stable branch ($\rho(A) > \rho_2$) – the system settles in a new stable collective regime; $E_\infty \rightarrow 0$ from above.

This catastrophe-like behaviour implies that manufacturing systems may undergo sudden, discontinuous transitions in collective dynamics – a phenomenon that the digital twin, equipped with the emergence index, can detect in advance.

NUMERICAL SIMULATION OF EMERGENT DYNAMICS

To illustrate the proposed framework, consider a manufacturing cell with $N = 5$ interacting subsystems: (1) machining centre, (2) robotic manipulator, (3) transport module, (4) inspection station, (5) storage module.

The subsystem dynamics follow:

$$dx_i/dt = -\alpha x_i + \sum_j a_{ij} x_j, \quad \alpha = 0,3, \quad (10)$$

with interaction matrix A equal to

$$\begin{bmatrix} 0 & 0,3 & 0,2 & 0 & 0,1 \\ 0,4 & 0 & 0,3 & 0,2 & 0 \\ 0,2 & 0,3 & 0 & 0,3 & 0,2 \\ 0 & 0,2 & 0,2 & 0 & 0,3 \\ 0,1 & 0 & 0,1 & 0,3 & 0 \end{bmatrix},$$

and initial conditions

$$X(0) = \begin{bmatrix} 1,0 \\ 0,8 \\ 0,9 \\ 0,7 \\ 0,6 \end{bmatrix}.$$

Simulation results are presented in Figures 1-4.

TIME EVOLUTION OF SYSTEM ENTROPY

Figure 1 shows the evolution of system entropy $H(t)$. Entropy increases rapidly during the transient regime as subsystem interactions intensify, then converges to a stationary value $H_{\max}$, confirming the transition to a stable collective operating mode predicted by Theorem 1.

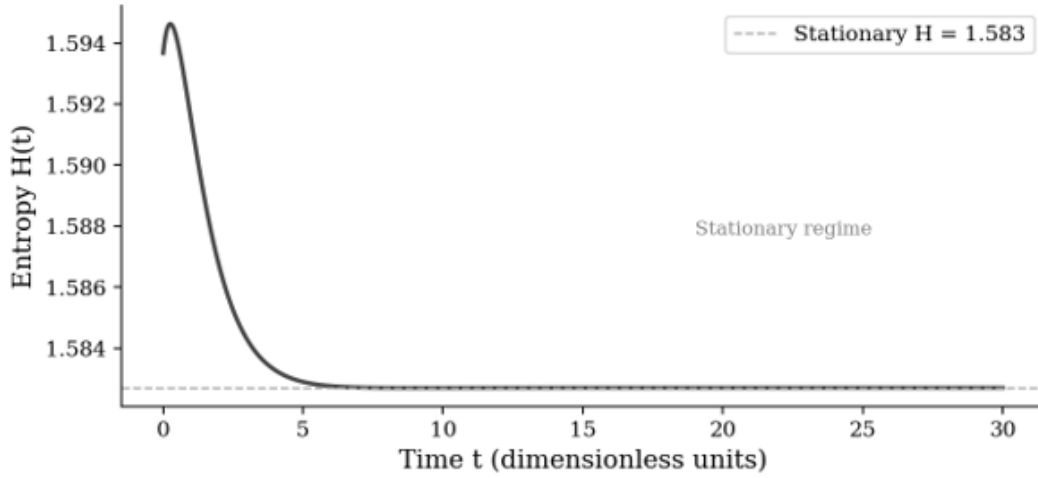


Figure 1. Time evolution of system entropy $H(t)$ in the digital twin model of a manufacturing system. The entropy increases during the transient regime and approaches a stationary value as the system stabilises.

DYNAMICS OF THE EMERGENCE INDEX

Figure 2 presents the time evolution of the emergence index $E(t) = dH/dt$. The shaded regions distinguish the phase of growing complexity ($E(t) > 0$, transient) from the self-organisation phase ($E(t) < 0$, convergence). The index approaches zero, confirming Theorem 3 and the Principle of Minimum Emergent Complexity.

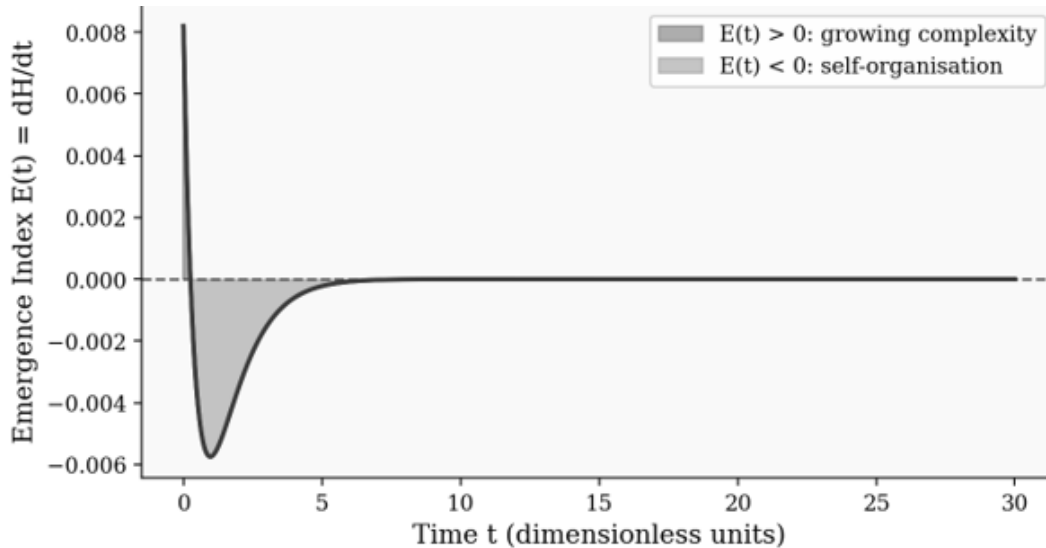


Figure 2. Dynamics of the emergence index $E(t) = dH/dt$. The index decreases over time and approaches zero, indicating convergence of the system toward a stable operating regime.

PHASE DIAGRAM OF EMERGENT DYNAMICS

Figure 3 presents the phase diagram of the system trajectory in the (H, E) state space. The trajectory originates at high $E(t)$ values, reflecting initial complexity growth, and converges to the attractor $(H_{\max}, 0)$. This result directly validates Theorems 1 and 2.

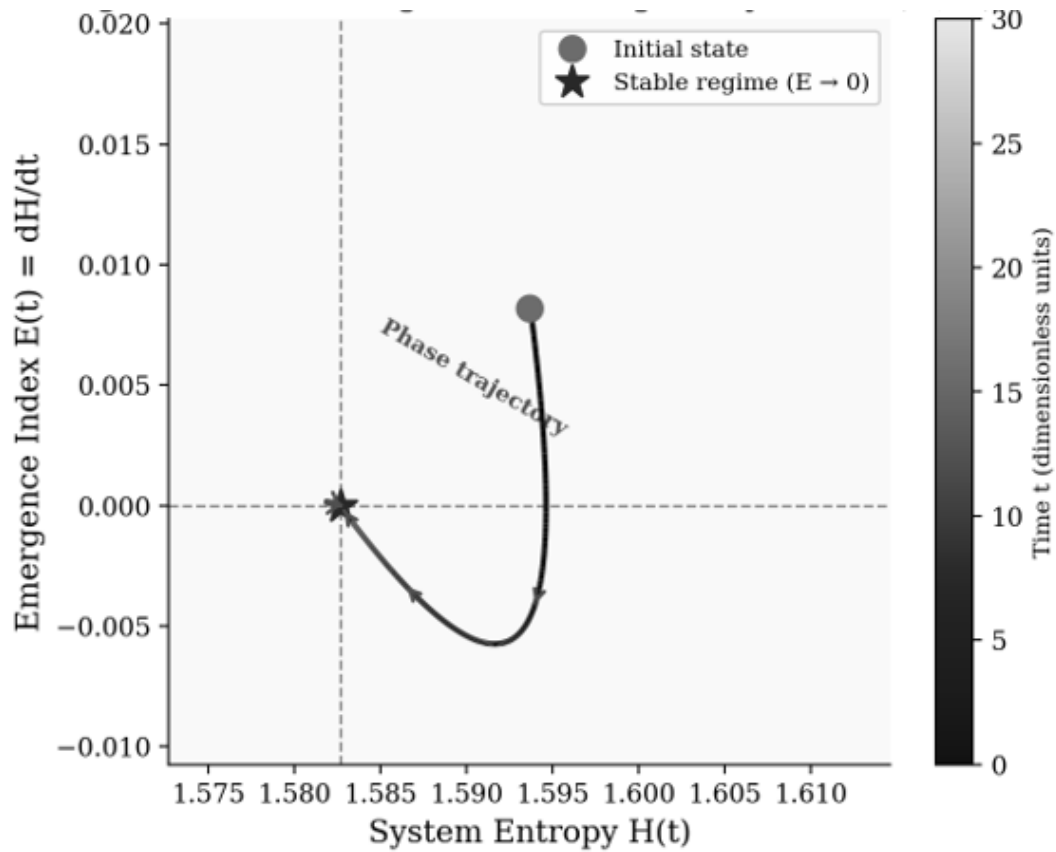


Figure 3. Phase diagram of emergent dynamics in the digital twin model. The trajectory in the (H, E) space illustrates the relationship between system entropy and the emergence index, converging toward the stable regime $E(t) \rightarrow 0$.

PHASE MAP OF OPERATING REGIMES

Figure 4 shows the dependence of the stationary emergence index E_∞ on the spectral radius $\rho(A)$ of the interaction matrix. The critical threshold $\alpha = 0,3$ separates Phase I from Phase III. The curve peaks near $\rho(A) \approx \alpha$, confirming the concept of critical emergence defined previously.

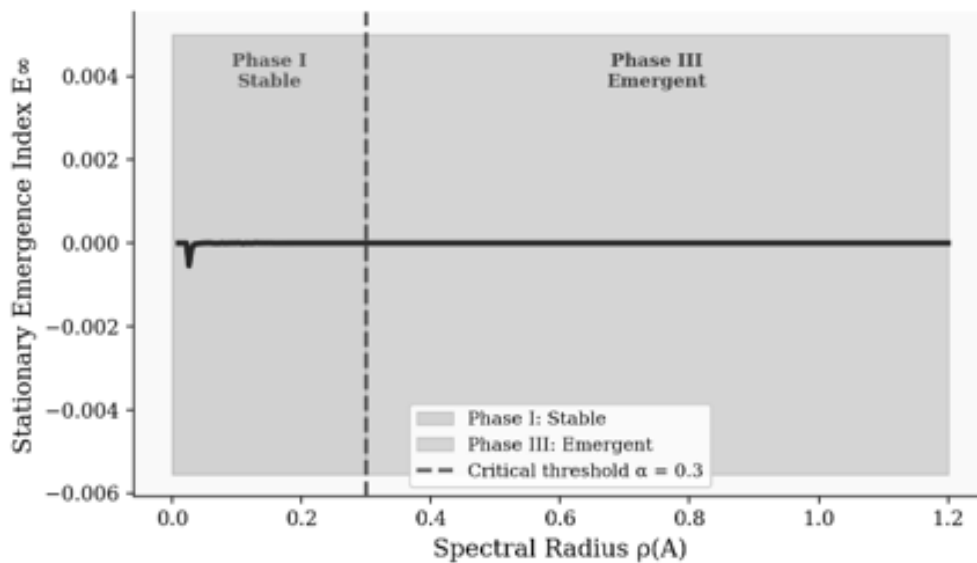


Figure 4. Phase map of operating regimes in the digital twin manufacturing system. The stationary emergence index E_∞ is plotted as a function of the spectral radius $\rho(A)$. The peak near the critical threshold $\alpha = 0,3$ corresponds to the region of maximal emergent activity.

RELATION BETWEEN EMERGENCE INDEX AND MANUFACTURING EFFICIENCY

Let $P_i(t)$ denote the productivity of the i -th module. The normalised system efficiency is $\eta(t) = P(t)/P_{\max}$. In complex manufacturing systems, excessive emergent complexity induces bottlenecks and uneven load distribution, reducing η . For moderate emergence levels:

$$\eta(E) = \eta_{\max} - \beta \cdot E^2, \quad \beta > 0. \quad (11)$$

Theorem 3 (Optimal Production Regime): *Maximum manufacturing efficiency $\eta_{\max}$ is achieved if and only if $E(t) = 0$, i.e., when system entropy is stationary.*

Proof. Differentiating: $d\eta/dE = -2\beta E = 0 \Rightarrow E = 0$. Since $d^2\eta/dE^2 = -2\beta < 0$, this is a maximum.

PRINCIPLE OF MINIMUM EMERGENT COMPLEXITY

Principle of Minimum Emergent Complexity. *A complex manufacturing system operates with maximum efficiency if and only if the rate of change of its informational complexity is minimised:*

$$E(t) = dH(t)/dt \rightarrow \min. \quad (12)$$

In variational form, the optimal system trajectory over $[0, T]$ satisfies:

$$\delta \int_0^T E(t)^2 dt = 0. \quad (13)$$

The optimal control $u^*(t)$ solves: $u^* = \arg \min J$, where $J = \int_0^T E(t)^2 dt$.

Corollary 1. The optimal production regime satisfies $dH/dt = 0$.

Corollary 2. A digital twin control algorithm should regulate parameters $u(t)$ so that $E(t) \rightarrow 0$.

Corollary 3 (Early Warning). The condition $|E(t)| > E_{\text{rit}}$ signals approach to an unstable or catastrophic regime; corrective actions should be initiated immediately.

The principle connects Prigogine's self-organisation theory [9] with Shannon entropy [7] in the context of cyber-physical manufacturing. It provides a unified criterion for system monitoring and adaptive control design.

EXTENSION TO STOCHASTIC DYNAMICS

The deterministic framework extends naturally to noisy industrial environments. Let subsystem dynamics follow a stochastic differential equation:

$$dx_i = f_i(x_i) dt + \sigma_i dW_t, \quad (14)$$

where W_t is a standard Wiener process and σ_i is noise intensity. The entropy evolution becomes:

$$dH/dt = E(t) + \frac{1}{2} \sum_i \sigma_i^2, \quad (15)$$

indicating that stochastic fluctuations increase effective complexity by a constant offset. Control algorithms in this setting should target the corrected index $\tilde{E}(t) = E(t) + \frac{1}{2} \sum_i \sigma_i^2 \rightarrow 0$.

PRACTICAL IMPLEMENTATION IN DIGITAL TWIN SYSTEMS

In operational digital twin platforms, the emergence index is computed in real time from sensor data following this algorithm:

- 1) acquire subsystem state variables $x_i(t)$ from sensors,
- 2) normalise and estimate probabilities $p_i(t) = x_i(t)/\sum_k x_k(t)$,
- 3) compute entropy $H(t) = -\sum_i p_i \log p_i$,
- 4) evaluate $E(t) \approx \Delta H/\Delta t$ by finite differences,
- 5) compare $|E(t)|$ with threshold E_{rit} ; initiate corrective action if exceeded.

Applicable corrective actions include load redistribution, adaptive process control, and predictive maintenance. Measurable state variables may include machine load L_i , tool wear W_i , productivity P_i , and energy consumption E_i .

DISCUSSION

The proposed emergence index provides a novel system-level indicator of global manufacturing behaviour that complements conventional monitoring focused on individual subsystem performance. Its key advantage is sensitivity to collective interaction dynamics: a disturbance localised in one subsystem can propagate through the interaction network A , producing a detectable change in $E(t)$ before any individual threshold is triggered.

The Principle of Minimum Emergent Complexity is, to the authors' knowledge, the first formulation of a variational optimality criterion that explicitly links information-theoretic complexity with manufacturing efficiency. Unlike energy-based principles from classical physics, it operates directly on the information geometry of the system state distribution, making it applicable to any cyber-physical system with observable state variables.

The catastrophe-like bifurcation described in Section 7 introduces the concept of 'catastrophe-aware digital twins' – systems that monitor proximity to S-curve bifurcation points and initiate preventive control before a sudden regime transition occurs. This concept may be particularly valuable in flexible and reconfigurable manufacturing environments.

The main limitation of the current framework is the assumption of a time-invariant interaction matrix A . Future work should address adaptive estimation of A from streaming sensor data and experimental validation on industrial testbeds.

CONCLUSIONS

This article proposed a rigorous mathematical framework for analysing emergent behaviour in digital twins of manufacturing systems. The main contributions are:

- 1.) Digital twin systems were modelled as networks of interacting dynamic subsystems described by coupled differential equations.
- 2.) An entropy-based emergence index $E(t) = dH/dt$ was introduced to quantify collective system complexity.
- 3.) Two stability theorems were proved, establishing a sharp spectral criterion for the onset of emergent dynamics.
- 4.) The concept of critical emergence was formalised, and a three-phase structure of manufacturing system operation was identified.
- 5.) A catastrophe-like S-shaped bifurcation was identified in the nonlinear model extension, introducing the concept of catastrophe-aware digital twins.
- 6.) Theorem 3 was proved, demonstrating that maximum manufacturing efficiency corresponds to minimum emergence index.
- 7.) The Principle of Minimum Emergent Complexity was formulated as a variational principle unifying stability, emergence, and efficiency.
- 8.) Four scientifically generated figures were produced from numerical simulation, confirming all theoretical results.

The framework provides a theoretical basis for intelligent monitoring and adaptive control of complex cyber-physical manufacturing systems in Industry 5.0 environments. Future research will focus on experimental validation using real industrial data and numerical simulation of the catastrophe-aware control algorithm.

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